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# Computing Automorphism Groups of Shifts using Atypical Equivalence Classes

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**Abstract:** We study the automorphism group of an infinite minimal shift  $(X, \sigma)$  such that the complexity difference function,  $p(n+1) - p(n)$ , is bounded. We give some new bounds on  $\text{Aut}(X, \sigma)/\langle \sigma \rangle$  and also study the one-sided case. For a class of Toeplitz shifts, including the class of shifts defined by constant-length primitive substitutions with a coincidence, and with height one, we show that the two-sided automorphism group is a cyclic group. We next focus on shifts generated by primitive constant-length substitutions. For these shifts, we give an algorithm that computes their two-sided automorphism group. Finally we show that with the same techniques, we are able to compute the set of conjugacies between two such shifts.

**Key words and phrases:** substitution dynamical systems, endomorphisms

## 1 Introduction

In this article we study the automorphism groups  $\text{Aut}(X, \sigma)$  of some “small” shifts  $(X, \sigma)$ . All terms are defined as they are needed in Sections 2 to 3.

In the brief Section 2, we show that for shifts with sublinear complexity, there are bounds on the orders of  $\text{Aut}(X, \sigma)$  for one-sided shifts and  $\text{Aut}(X, \sigma)/\langle \sigma \rangle$  for two-sided shifts.

In [3], Coven explicitly describes the endomorphism monoid of any non-trivial two-sided constant-length substitution shift  $(X, \sigma)$  on two letters. Either the shift has a metric discrete spectrum, in which case  $\text{Aut}(X, \sigma)$  consists of the powers of the shift, or the shift has a partly continuous spectrum, in which case the “letter-exchanging” automorphism is also present. In both cases all endomorphisms are automorphisms. For constant-length substitutions on larger alphabets, those whose shifts have a discrete spectrum are precisely those that have *coincidences*, [9, Thm 7] and all other constant-length shifts,

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including the family of *bijective substitutions*, have a partly continuous spectrum. (See Section 3.3 for the definitions of these two terms.)

Primitive substitution shifts are uniquely ergodic [21], and in the measurable setting Lemanczyk and Mentzen [19] show that any isomorphism of a bijective substitution is the composition of a shift with a letter-to-letter code. Host and Parreau [17] extend the results of [19], showing that for the constant-length substitution shifts which do not have a purely discrete spectrum, maps in the commutant<sup>1</sup> are essentially topological and, up to a shift, have small radius.

Substitution shifts arising from primitive substitutions are a sub-family of shifts with sublinear complexity. We use a non-trivial result of Cassaigne [2] to prove Theorem 2.3, which tells us that for infinite minimal one-sided shifts  $(X, \sigma)$  with sublinear complexity, automorphisms have a bounded order, and, modulo powers of the shift, the same is true for infinite minimal two-sided shifts with sublinear complexity. There are more general results, with similar proofs, in recent articles, by Donoso, Durand, Maass and Petite in [10, Theorem 1.1], and Cyr and Kra in [8, Theorem 1.4], both for the two-sided case. In both of these articles, the condition of sublinear complexity is relaxed ( $\liminf p_X(n)/n$ , rather than  $\limsup p_X(n)/n$ , is required to be finite). Moreover, Cyr and Kra generalise this theorem to non-minimal shifts. We prove a related one-sided result. Also, under the assumption of sublinear complexity, our bound gives sharper bounds in some cases, but requires finer information about the branch points or asymptotic orbits. We give a case where the bound is tighter in Example 2.4. We also mention Olli's work [23], and also Salo and Törmä's [24].

In Section 2.3, we use results in the literature to give bounds on the orders of elements in  $\text{Aut}(X, \sigma)/\langle \sigma \rangle$  for known families of shifts with sublinear complexity. We also formulate a notion of one-sided coalescence in Theorem 2.5: Example 3.23 tell us that not every endomorphism of a one-sided shift is an automorphism composed with a shift.

The proof of Theorem 2.3 is not constructive. Indeed, for an arbitrary shift with sublinear complexity, it does not seem to be meaningful to ask for an algorithm that computes its automorphism group: it is not clear what sort of data it would take as input. Nevertheless the question of whether there exists an algorithm to compute  $\text{Aut}(X, \sigma)$  makes sense for substitution shifts. In Section 3, we focus on this problem for the family of primitive constant length  $r$  substitutions. There we assume that our substitution  $\theta$  is primitive, constant-length, and generates an infinite shift.

We reprove Theorem 2.3 for these shifts, obtaining finer information. Our proof techniques are a generalisation of Coven's strategy in [3] for constant-length substitutions on a two-letter alphabet. There, the fact that these shifts are a somewhere one-to-one extension of their maximal equicontinuous factor  $\mathbb{Z}_r$  was used to associate to each automorphism a "fingerprint" in  $\mathbb{Z}_r$  which must satisfy certain constraints: we re-state this technique in Theorem 3.3.

We show that for the case of larger alphabets a similar strategy works. Having transferred our search to  $\mathbb{Z}_r$ , we prove in Lemma 3.14 that the constraints mentioned above imply that the fingerprints of elements of  $\text{Aut}(X_\theta, \sigma)$  are rational numbers with small denominators. We use the arithmetic properties of odometers to show in Corollary 3.7 and the note that follows it, that the automorphism groups of a large family of Toeplitz shifts, including those generated by constant-length substitutions with a coincidence, with *height* one are cyclic. The same arithmetic properties imply that these shifts have trivial one-sided automorphism groups (Theorem 3.18). A technical detail is that not every rational that satisfies the

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<sup>1</sup>The *commutant* is the measurable analogue of the endomorphism semigroup.

desired constraints is the fingerprint of some automorphism in  $\text{Aut}(X_\theta, \sigma)$ . In Propositions 3.19 and 3.21, we find checkable conditions for the existence of an automorphism. Our arguments culminate in Theorem 3.22, where we describe an algorithm to compute  $\text{Aut}(X_\theta, \sigma)$ . We illustrate our constructive methods with some examples, including one where  $\langle \sigma \rangle \subsetneq \text{Aut}(X_\theta, \sigma)$ .

With some minor modifications, we can also describe in Section 3.7 an algorithm to compute the set of conjugacies between two shifts generated by constant-length substitutions. We note that Durand and Leroy [14] have communicated to us that they have a result for arbitrary primitive substitutions. Coven, Dekking and Keane [4] have an alternative proof of the decidability of topological conjugacy of constant-length substitution shifts. When we started the project, their decision procedure for testing conjugacy was shown to terminate for many, but not all constant-length substitutions. They have since proved that their procedure always terminates.

We remark that there is a common thread between the arguments in Section 2 and Section 3, namely that in both cases we study an equivalence relation on the points of  $X$  (asymptotic equivalence and having a common image in the maximal equicontinuous factor respectively). While most equivalence classes are small (of size 1 and  $c$ , the *column number*, respectively), the larger equivalence classes have to be mapped to themselves under automorphisms, leading to restrictions on the collection of possible automorphisms.

## 2 Automorphisms of minimal shifts with sublinear complexity

### 2.1 Notation

Let  $\mathcal{A}$  be a finite alphabet, with the discrete topology, and let  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ . We endow  $\mathcal{A}^{\mathbb{N}_0}$  and  $\mathcal{A}^{\mathbb{Z}}$  with the product topology, and let  $\sigma : \mathcal{A}^{\mathbb{N}_0} \rightarrow \mathcal{A}^{\mathbb{N}_0}$  (or  $\sigma : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ ) denote the shift map. We consider only infinite minimal shifts  $(X, \sigma)$ , which can be either one- or two-sided. An *endomorphism* of  $(X, \sigma)$  is a map  $\Phi : X \rightarrow X$  which is continuous, onto, and commutes with  $\sigma$ ; if in addition  $\Phi$  is one-to-one, then  $\Phi$  is called an *automorphism*. We say  $\Phi$  has *finite order*  $m$  if  $m$  is the least positive integer such that  $\Phi^m$  is the identity, and that  $\Phi$  is a *k-th root of a power of the shift* if there exists an  $n$  such that  $\Phi^k = \sigma^n$ . Let  $\text{Aut}(X, \sigma)$  denote the automorphism group of  $(X, \sigma)$ . If  $X$  is two-sided, then  $\text{Aut}(X, \sigma)$  contains the (normal) subgroup generated by the shift, denoted by  $\langle \sigma \rangle$ .

The *language* of a minimal shift  $(X, \sigma)$ , denoted  $\mathcal{L}_X$ , is the set of all finite words that occur in points of  $X$ . We denote by  $p(n)$ ,  $n \geq 1$ , the *complexity function* of  $(X, \sigma)$ :  $p(n)$  is the number of words (also called blocks) in  $\mathcal{L}_X$  of length  $n$ . A symbolic system  $(X, \sigma)$  has *sublinear complexity* if its complexity function is bounded by a linear function.

### 2.2 Automorphisms of minimal shifts with sublinear complexity

If  $(\bar{X}, \bar{\sigma})$  is a one-sided minimal shift, let  $(X, \sigma)$  denote the two-sided version of  $(\bar{X}, \bar{\sigma})$  i.e.  $X$  consists of all bi-infinite sequences whose finite subwords belong to  $\mathcal{L}_{\bar{X}}$ . Given a two-sided shift  $(X, \sigma)$  we define similarly its one-sided version. If  $(\bar{X}, \bar{\sigma})$  is minimal, then the one-sided version of  $(X, \sigma)$  is  $(\bar{X}, \bar{\sigma})$  itself. Henceforth  $(\bar{X}, \bar{\sigma})$  refers to a one-sided shift and  $(X, \sigma)$  refers to a two-sided shift.

For  $k > 1$ , we say that  $\bar{x} \in \bar{X}$  is a *branch point of order k* if  $|\sigma^{-1}(\bar{x})| = k$ . If  $(X, T)$  is a minimal invertible dynamical system, we define an equivalence relation  $\sim$  on orbits in  $X$  as follows: Two orbits  $\mathcal{O}_x = \{T^n(x) : n \in \mathbb{Z}\}$  and  $\mathcal{O}_y = \{T^n(y) : n \in \mathbb{Z}\}$  in  $X$  are *right asymptotic*, denoted  $\mathcal{O}_x \sim \mathcal{O}_y$ , if there exists  $m \in \mathbb{Z}$  such that  $d(T^{m+n}x, T^ny) \rightarrow 0$  as  $n \rightarrow \infty$ . The right asymptotic equivalence class of  $\mathcal{O}_x$  will be denoted by  $[x]$ . A right asymptotic equivalence class will be said to be non-trivial if it does not consist of a single orbit. Clearly in the case that  $X$  is a shift,  $\mathcal{O}_x$  and  $\mathcal{O}_y$  are right asymptotic if there exists  $m$  such that  $x_{m+n} = y_n$  for all  $n$  sufficiently large.

**Lemma 2.1.** *Let  $(\bar{X}, \bar{\sigma})$  be an infinite minimal one-sided shift and let  $(X, \sigma)$  be the corresponding two-sided shift. If  $(\bar{X}, \bar{\sigma})$  has sublinear complexity, then  $(\bar{X}, \bar{\sigma})$  has finitely many branch points, and  $\{\mathcal{O}_x : x \in X\}$  has finitely many non-trivial right asymptotic equivalence classes.*

*Proof.* We prove that if  $(\bar{X}, \bar{\sigma})$  has sublinear complexity, then  $(\bar{X}, \bar{\sigma})$  has finitely many branch points, as the latter implies that  $\{\mathcal{O}_x : x \in X\}$  has finitely many non-trivial right asymptotic equivalence classes.

Define for  $n \geq 1$  the *complexity difference function*  $s(n) := p(n+1) - p(n)$ . Since  $(\bar{X}, \bar{\sigma})$  has sublinear complexity,  $s(n)$  is bounded [2], say  $s(n) \leq L$  for all  $n \geq 1$ . Then there are at most  $L$  words of length  $n$  with at least two left extensions, and so there are at most  $L$  branch points in  $\bar{X}$ .  $\square$

Next we have a basic lemma about automorphisms of minimal (not necessarily symbolic) dynamical systems.

**Lemma 2.2.** *Let  $(X, T)$  be a minimal dynamical system.*

1. Suppose  $T$  is not invertible. If there is a point  $x \in X$  and a finite subset  $F$  of  $X$  such that  $\Phi(x) \in F$  for every automorphism  $\Phi$  of  $(X, T)$ , then  $|\text{Aut}(X, T)| \leq |F|$ . In particular, every automorphism has order at most  $|F|$ .
2. Suppose  $T$  is invertible. If there is a point  $x \in X$  and a finite collection  $\mathcal{F}$  of right asymptotic equivalence classes such that  $[\Phi(x)] \in \mathcal{F}$  for every automorphism  $\Phi$  of  $(X, T)$ , then  $|\text{Aut}(X, T)/\langle T \rangle| \leq |\mathcal{F}|$ . In particular, the order of each element of  $\text{Aut}(X, T)/\langle T \rangle$  is at most  $|\mathcal{F}|$ .

*Proof.* For the first part, notice that an automorphism of a minimal dynamical system is determined by its action on a single point.

For the second part, suppose that  $\Phi$  and  $\Psi$  are automorphisms of  $(X, T)$  such that  $\mathcal{O}_{\Phi(x)} \sim \mathcal{O}_{\Psi(x)}$  for some  $x$ . Let  $m$  be such that  $d(T^n(\Phi(x)), T^{n+m}(\Psi(x))) \rightarrow 0$ . Then we claim that  $\Phi = \Psi \circ T^m$ . For any  $x' \in X$ , let  $(n_i)$  be an increasing sequence of integers so that  $T^{n_i}x \rightarrow x'$ . Now we have

$$\begin{aligned} \Phi(x') &= \lim_{i \rightarrow \infty} \Phi(T^{n_i}x) = \lim_{i \rightarrow \infty} T^{n_i}\Phi(x) \\ &= \lim_{i \rightarrow \infty} T^{n_i+m}\Psi(x) = \lim_{i \rightarrow \infty} \Psi(T^m(T^{n_i}x)) = \Psi(T^mx'), \end{aligned}$$

where for the third equality, we used that fact that  $\mathcal{O}_{\Phi(x)}$  and  $\mathcal{O}_{\Psi(x)}$  are right asymptotic. Hence we deduce that an automorphism of  $X$  is determined up to composition with a power of  $T$  by the right asymptotic equivalence class of the image of a single point.  $\square$

The following result tells us that for infinite minimal shifts with sublinear complexity,  $|\text{Aut}(X, \sigma)/\langle \sigma \rangle|$  is bounded by the number of maximal sets of mutually right asymptotic orbits.

**Theorem 2.3.** *Let  $(\bar{X}, \bar{\sigma})$  be an infinite minimal one-sided shift and let  $(X, \sigma)$  be the corresponding two-sided shift. For  $k > 1$ , let  $\bar{M}_k$  be the number of  $k$ -branch points in  $\bar{X}$ , and  $M_k$  be the number of  $\sim$ -equivalence classes of size  $k$  in  $X$ . Suppose that  $(\bar{X}, \bar{\sigma})$  has sublinear complexity, so that both  $\bar{M}_k$  and  $M_k$  are finite. Then*

1.  $\text{Aut}(\bar{X}, \bar{\sigma})$  has at most  $\bar{M} := \min\{\bar{M}_k : \bar{M}_k > 0\}$  elements, and
2.  $\text{Aut}(X, \sigma)/\langle \sigma \rangle$  has at most  $M := \min\{M_k : M_k > 0\}$  elements.

*Proof.* Using Lemma 2.1,  $\bar{X}$  has finitely many branch points and  $\{\mathcal{O}_x : x \in X\}$  has finitely many non-trivial right asymptotic equivalence classes. The fact that the systems are infinite implies that  $\bar{X}$  (respectively  $X$ ) has at least one branch point (respectively one non-trivial right asymptotic equivalence class) so that  $\bar{M}$  and  $M$  are positive and finite.

Let  $\bar{M} = \bar{M}_k$ . Let  $\bar{x}$  be a branch point of order  $k$ . An automorphism  $\bar{\Phi}$  of  $(\bar{X}, \bar{\sigma})$  must map branch points of order  $k$  to branch points of order  $k$ , so we apply Lemma 2.2(1), to the set of branch points of order  $k$ , to obtain (1).

Similarly, let  $M = M_\ell$  and let  $x$  be such that  $\mathcal{O}_x$  belongs to an equivalence class consisting of  $\ell$  orbits. If  $\Phi$  is an automorphism  $(X, \sigma)$ , then  $\mathcal{O}_{\Phi(x)}$  must also belong to an equivalence class consisting of  $\ell$  orbits. Hence by Lemma 2.2(2), there are at most  $M_\ell$  automorphisms of  $(X, \sigma)$  up to composition with a power of  $\sigma$ .  $\square$

As  $\text{Aut}(\bar{X}, \bar{\sigma})$  and  $\text{Aut}(X, \sigma)/\langle \sigma \rangle$  are both finite groups, the order of any element of  $\text{Aut}(\bar{X}, \bar{\sigma})$  divides  $|\text{Aut}(\bar{X}, \bar{\sigma})|$ , and any element of  $\text{Aut}(X, \sigma)$  is a  $k$ -th root of the shift, where  $k$  divides  $|\text{Aut}(X, \sigma)/\langle \sigma \rangle|$ .

## 2.3 Examples and bounds

### 2.3.1 Substitution shifts.

A *substitution* is a map from  $\mathcal{A}$  to the set of nonempty finite words on  $\mathcal{A}$ . We use concatenation to extend  $\theta$  to a map on finite and infinite words from  $\mathcal{A}$ . We say that  $\theta$  is *primitive* if there is some  $k \in \mathbb{N}$  such that for any  $a, a' \in \mathcal{A}$ , the word  $\theta^k(a)$  contains at least one occurrence of  $a'$ . By iterating  $\theta$  on any fixed letter in  $\mathcal{A}$ , we obtain one-sided (right) infinite points  $u = u_0 \dots$  such that  $\theta^j(u) = u$  for some natural  $j$ . The pigeonhole principle implies that  $\theta$ -periodic points always exist, and, for primitive substitutions, we define  $\bar{X}_\theta$  to be the shift orbit closure of any one of these  $\theta$ -periodic points and call  $(\bar{X}_\theta, \bar{\sigma})$  a *one-sided substitution shift*. Barge, Diamond and Holton [1] show that a primitive substitution on  $k$  letters has at most  $k^2$  right asymptotic orbits, so that we can apply Theorem 2.3 to deduce that any automorphism of  $(X_\theta, \sigma)$  is a  $j$ -th root of a power the shift for some  $j \leq k^2$ .

**Example 2.4.** Let  $\mathcal{A} = \{a_n, b_n, c_n : 1 \leq n \leq N\}$ . Let  $W$  be a word which contains all of the letters in  $\mathcal{A}$ . Define a substitution,  $\theta$ , by  $\theta(a_n) = Wa_n a_n$ ,  $\theta(b_n) = Wb_n a_n$  and  $\theta(c_n) = Wc_n a_n$  for  $1 \leq n \leq N$ . Now the one-sided shift  $(\bar{X}_\theta, \bar{\sigma})$  contains  $N$  branch points of order three,  $\{x^n : 1 \leq n \leq N\}$ , where  $x^n$  satisfies the equation  $a_n \theta(x^n) = x^n$ . Also,  $\bar{X}_\theta$  contains only one other branch point of order  $N$ : the unique right-infinite  $\theta$ -fixed point. Similarly, in the two-sided shift  $(X_\theta, \sigma)$ , there are exactly  $N$  distinct right asymptotic equivalence classes of size 3, and one right asymptotic equivalence class of size  $N$ . Now our bounds from Theorem 2.3 tell us that each of  $\text{Aut}(\bar{X}_\theta, \bar{\sigma})$  and  $\text{Aut}(X_\theta, \sigma)/\langle \sigma \rangle$  consist of one element.

We compare our bounds to those that would be obtained using the results in [8] or [10]. In the latter, Theorem 3.1 tells us that  $|\text{Aut}(X)/\langle \sigma \rangle|$  divides  $3N + 1$ , the number of non-trivial right asymptotic equivalence classes. In the former, Theorem 1.4 tells us that if  $\liminf p(n)/n < k$ , then  $|\text{Aut}(X)/\langle \sigma \rangle| < k$ . If one takes the word  $W = a_1 b_1 c_1 a_2 b_2 c_2 \dots a_N b_N c_N$ , a crude check shows that  $\liminf p(n)/n \geq 3N/2$ , so that the upper bound for  $|\text{Aut}(X)/\langle \sigma \rangle|$  coming from Theorem 1.4 of [8] would be at least  $3N/2$ .

### 2.3.2 Linearly recurrent shifts

The notions below are the same for one-sided and two-sided shifts, so we will state them only for two-sided shifts, thus avoiding the bars. If  $u, w \in \mathcal{L}_X$ , we say that  $w$  is a *return word to  $u$*  if (a)  $u$  is a prefix of  $w$ , (b)  $wu \in \mathcal{L}_X$ , and (c) there are exactly two occurrences of  $u$  in  $wu$ . Letting  $\ell(u)$  denote the length of  $u$ , a minimal shift  $(X, \sigma)$  is *linearly recurrent* if there exists a constant  $K$  such that for any word  $u \in \mathcal{L}_X$  and any return word (to  $u$ )  $w$ ,  $\ell(w) \leq K\ell(u)$ . Such a  $K$  is called a *recurrence constant*. Durand, Host and Skau show that if  $(X, \sigma)$  has linear recurrence constant  $K$ , then its complexity is bounded above by  $Kn$  for large  $n$  [13, Theorem 23]. Cassaigne [2] shows that if  $p(n) \leq Kn + 1$ , then there are at most  $2K(2K + 1)^2$  branch points and asymptotic orbits, and we can now apply Theorem 2.3 to bound  $|\text{Aut}(\bar{X}, \bar{\sigma})|$  and  $|\text{Aut}(X, \sigma)/\langle \sigma \rangle|$  above by  $2(K + 1)(2K + 3)^2$ .

Durand [12, Corollary 18] shows that for linearly recurrent two-sided shifts, any endomorphism is an automorphism. The corresponding one-sided version of this is:

**Theorem 2.5.** Let  $(\bar{X}, \bar{\sigma})$  be an infinite, minimal, linearly recurrent one-sided shift with recurrence constant  $K$ . Then every endomorphism of  $(\bar{X}, \bar{\sigma})$  is a  $k$ -th root of a power of the shift for some positive  $k \leq 2(K + 1)(2K + 3)^2$ .



*Proof.* Any endomorphism  $\bar{\Phi}$  of  $(\bar{X}, \bar{\sigma})$  defines an endomorphism  $\Phi$  of  $(X, \sigma)$ , which must be an automorphism by [12, Corollary 18]. By Cassaigne's result,  $\Phi^k = \sigma^n$  for some positive  $k \leq 2(K+1)(2K+3)^2$  and some  $n \in \mathbb{Z}$ , so  $\bar{\Phi}^k = \bar{\sigma}^n$ .  $\square$

In Example 3.23 we describe a non-trivial endomorphism of  $(\bar{X}, \bar{\sigma})$  which is not a power of the shift.

### 3 Constant length substitutions

Let  $\theta$  be a substitution on the alphabet  $\mathcal{A}$ . The substitution  $\theta$  has (constant) length  $r$  if for each  $a \in \mathcal{A}$ ,  $\theta(a)$  is a word of length  $r$ . In this section we generalise the results of Coven in [3] to primitive constant-length substitutions on a finite alphabet. We then provide algorithms to compute the automorphism group of a constant-length substitution shift, and also the set of conjugacies between two constant-length substitution shifts.

#### 3.1 The $r$ -adic integers

Let  $\mathbb{Z}_r$  denote the set of  $r$ -adic integers, identified with the one-sided shift space consisting of sequences  $(x_n)_{n \geq 0}$  with the  $x_n$ 's taking values in  $\{0, 1, \dots, r-1\}$ . We think of these expansions as being written from right to left, so that  $\mathbb{Z}_r$  consists of left-infinite sequences of digits,  $x_0$  is the rightmost digit of  $x$ , and so that addition in  $\mathbb{Z}_r$  has the carries propagating to the left in the usual way. Formally  $\mathbb{Z}_r$  is the inverse limit of rings  $\mathbb{Z}/r^n\mathbb{Z}$ , and is itself a ring.

We make no assumption about the primality of  $r$ . If  $r$  is not a prime power, then  $\mathbb{Z}_r$  has zero divisors. Nevertheless, if  $k \in \mathbb{Z}$  and  $\gcd(k, r) = 1$ , then multiplication by  $k$  is an isomorphism of  $\mathbb{Z}_r$  as an additive group, so that one can make sense of expressions such as  $m/k$  for  $m \in \mathbb{Z}_r$  if  $\gcd(k, r) = 1$ . If  $k \in \mathbb{Z}$  and  $\gcd(k, r) > 1$ , then multiplication by  $k$  is still an injection of  $\mathbb{Z}_r$ . A version of the standard argument shows that if  $m$  and  $k$  are integers with  $\gcd(k, r) = 1$ , then  $m/k$  has an eventually periodic digit sequence; and conversely any point of  $\mathbb{Z}_r$  with an eventually periodic digit sequence can be written in the form  $m/k$  for  $m$  and  $k$  in  $\mathbb{Z}$  with  $\gcd(k, r) = 1$ . We may naturally think of  $\mathbb{Z}$  as a subset of  $\mathbb{Z}_r$ : a point  $s$  in  $\mathbb{Z}_r$  whose digits are eventually all 0's or eventually all  $(r-1)$ 's is the  $r$ -adic expansion of an integer: we write  $s \in \mathbb{Z}$ . Note also that as an additive group,  $\mathbb{Z}_r$  is torsion free.

We need the following fact.

**Lemma 3.1.** *Let  $kt \in \mathbb{Z}$  for some  $t \in \mathbb{Z}_r$  and  $k \in \mathbb{Z} \setminus \{0\}$ . Then there exists  $m|k$  such that  $mt \in \mathbb{Z}$  and  $\gcd(m, r) = 1$ .*

*Proof.* Let  $k = lm$ , where  $\gcd(m, r) = 1$  and  $l|r^n$ . Let  $p := r^n/l$  and  $s := mt$ . Now  $r^n s = pkt \in \mathbb{Z}$ . However,  $r^n s \in r^n \mathbb{Z}_r$ , which is the set of elements of  $\mathbb{Z}_r$  with at least  $n$  trailing 0's. Dividing by  $r^n$  removes these 0's, showing that  $s \in \mathbb{Z}$  as required.  $\square$

**Lemma 3.2.** *Let  $H$  be a subgroup of  $\mathbb{Z}_r$  containing  $\mathbb{Z}$  generated by a collection of elements  $(p_i/q_i)_{i=1}^n$  with  $\gcd(p_i, q_i) = \gcd(q_i, r) = 1$ . Then  $H$  is generated by a single element of the form  $1/q$ .*

*Proof.* Let  $\alpha_i = p_i/q_i$  and  $q = \text{lcm}(q_1, \dots, q_n)$ , so that if  $\tilde{p}_i := p_i(q/q_i)$  we have  $q\alpha_i = \tilde{p}_i$  for  $i = 1, \dots, n$ . Note that if  $m$  is a prime and  $m^\alpha | q$ , with  $\alpha$  maximal, then  $m^\alpha | q_i$  for some  $i$ , so that  $m \nmid p_i$  and  $m \nmid \tilde{p}_i$ . Hence



$\gcd(q, \tilde{p}_1, \dots, \tilde{p}_n) = 1$  and there exist integers  $k_0, k_1, \dots, k_n$  such that  $k_0q + k_1\tilde{p}_1 + \dots + k_n\tilde{p}_n = 1$ . Then  $k_0 + k_1\alpha_1 + \dots + k_n\alpha_n = 1/q \in H$  and all of the generators are multiples of  $1/q$ , so that  $H$  is generated by  $1/q$ .  $\square$

### 3.2 Maximal equicontinuous factors

If  $(X, T)$  is a continuous dynamical system, we say that the dynamical system  $(Y, S)$  is the *maximal equicontinuous factor* of  $(X, T)$  if  $(Y, S)$  is an equicontinuous factor of  $(X, T)$  with the property that any other equicontinuous factor  $(Z, R)$  of  $(X, T)$  is a factor of  $(Y, S)$ . The maximal equicontinuous factor of a minimal transformation is a rotation on a compact monothetic topological group, that is a group  $G$  for which there exists an element  $a$  such that the subgroup generated by  $a$  is dense. Such a group is always abelian [11] and we will write the group operation additively.

The systems we consider will have maximal equicontinuous factor  $(\mathbb{Z}_r, +1)$  or  $(\mathbb{Z}_r \times \{0, \dots, h-1\}, +1)$ , where the group operation on  $\mathbb{Z}_r \times \{0, \dots, h-1\}$  is

$$(x, i) + (y, j) = \begin{cases} (x+y, i+j) & \text{if } i+j < h; \\ (x+y+1, i+j-h) & \text{otherwise,} \end{cases}$$

and where  $r$  is not necessarily prime.

Let  $\text{End}(X, \sigma)$  be the set of endomorphisms of  $(X, \sigma)$ . Given two shifts  $(X, \sigma)$  and  $(Y, \sigma)$ , let  $\text{Conj}(X, Y)$  denote the (possibly empty) set of topological conjugacies between  $(X, \sigma)$  and  $(Y, \sigma)$ , and let  $\text{Fac}(X, Y)$  denote the set of factor maps from  $(X, \sigma)$  to  $(Y, \sigma)$ . For completeness we prove the following, which is a straightforward generalization of work done by Coven in [3, §3].

**Theorem 3.3.** *Let  $(X, \sigma)$  and  $(Y, \sigma)$  be infinite minimal shifts. Suppose that the group rotation  $(G, R)$  is the maximal equicontinuous factor of both  $(X, \sigma)$  and  $(Y, \sigma)$  and let  $\pi_X$  and  $\pi_Y$  be the respective factor maps. Then there is a map  $\kappa : \text{Fac}(X, Y) \rightarrow G$  such that*

$$\pi_Y(\Phi(x)) = \kappa(\Phi) + \pi_X(x)$$

for all  $x \in X$  and  $\Phi \in \text{Fac}(X, Y)$ . Also

1. if  $(Z, \sigma)$  is another shift which satisfies the assumptions on  $(X, \sigma)$ , then  $\kappa(\Psi \circ \Phi) = \kappa(\Psi) + \kappa(\Phi)$  for  $\Phi \in \text{Fac}(X, Y)$ ,  $\Psi \in \text{Fac}(Y, Z)$ , and
2. if  $\min_{g \in G} |\pi_X^{-1}(g)| = \min_{g \in G} |\pi_Y^{-1}(g)| = c < \infty$ , then
  - (a) for each  $\Phi \in \text{Fac}(X, Y)$ , we have

$$\{z \in \mathbb{Z}_r : |\pi_Y^{-1}(z)| > c\} \subset \{z \in \mathbb{Z}_r : |\pi_X^{-1}(z)| > c\} + \kappa(\Phi),$$

and

- (b)  $\kappa$  is at most  $c$ -to-one. In particular, if  $\pi_X$  and  $\pi_Y$  are somewhere one-to-one, then  $\kappa$  is an injection.

*Proof.* First we show that given any factor map  $\Phi : X \rightarrow Y$ , there exists  $\kappa(\Phi) \in \mathbb{Z}_r$  such that  $\pi_X(x) + \kappa(\Phi) = \pi_Y(\Phi(x))$ . Fix any  $b \in X$ , and define  $\kappa(\Phi) := \pi_Y(\Phi(b)) - \pi_X(b)$ . Notice that  $f(x) = \pi_Y(\Phi(x)) - \pi_X(x)$  is a  $T$ -invariant continuous function. Since  $T : X \rightarrow X$  is minimal, it follows that  $f(x) = f(b) = \kappa(\Phi)$  for all  $x \in X$ .

Thus for each factor mapping  $\Phi : (X, \sigma) \rightarrow (Y, \sigma)$  there is a map  $F : G \rightarrow G$ , namely  $F(g) := g + \kappa(\Phi)$ , such that  $\pi_Y \circ \Phi = F \circ \pi_X$ . This gives us a map  $\kappa : \text{Fac}(X, Y) \rightarrow G$ .

With the assumptions of (1), note that  $\pi_Z(\Psi \circ \Phi(x)) = \kappa(\Psi \circ \Phi) + \pi_X(x)$ , but also  $\pi_Z(\Psi \circ \Phi(x)) = \kappa(\Psi) + \pi_Y(\Phi(x)) = \kappa(\Psi) + \kappa(\Phi) + \pi_X(x)$ , and (1) follows.

Suppose that  $\Phi \in \text{Fac}(X, Y)$ . Let  $|\pi_Y^{-1}(z)| > c$ , so that there exist distinct  $y_1, \dots, y_{c+1}$  in  $Y$  with  $\pi_Y(y_i) = z$  for each  $i$ . Since  $\Phi$  is surjective, there exist  $x_1, \dots, x_{c+1}$  with  $\Phi(x_i) = y_i$ , and we have  $\pi_X(x_i) = z - \kappa(\Phi)$  for each  $x_i$ . Hence  $|\pi_X^{-1}(z - \kappa(\Phi))| > c$ , so  $z \in \{z' \in \mathbb{Z}_r : |\pi_X^{-1}(z')| > c\} + \kappa(\Phi)$ .

Let  $t$  be in the range of  $\kappa$ , let  $z \in \mathbb{Z}_r$  satisfy  $|\pi_Y^{-1}(z)| = c$  and let  $x \in \pi_X^{-1}(z - t)$ . Then if  $\kappa(\Phi) = t$ , we have  $\Phi(x) \in \pi_Y^{-1}(z)$ . Since a factor map of a minimal system is determined by its action on a single point, we deduce that there are at most  $c$  factor maps in  $\text{Fac}(X, Y)$ , proving (2b).  $\square$

### 3.3 Background results on constant-length substitutions

In this section, we collect some results that we use later to reduce the case of general constant-length substitutions to cases that can be more straightforwardly handled.

Let  $\theta$  be a primitive length  $r$  substitution with fixed point  $u$ , and with  $(X_\theta, \sigma)$  infinite. The *height*  $h = h(\theta)$  of  $\theta$  is defined as

$$h(\theta) := \max\{n \geq 1 : \gcd(n, r) = 1, n \mid \gcd\{a : u_a = u_0\}\}.$$

If  $h > 1$ , this means that  $\mathcal{A}$  decomposes into  $h$  disjoint subsets:  $\mathcal{A}_1 \cup \dots \cup \mathcal{A}_h$ , where a symbol from  $\mathcal{A}_i$  is always followed by a symbol from  $\mathcal{A}_{i+1}$ . Such a system is a constant height suspension. In fact, as shown in [9, Remark 9, Lemmas 17 and 19], such a system is conjugate to a constant height suspension of another constant-length substitution.

**Proposition 3.4.** *Let  $\theta$  be a primitive, length  $r$  substitution defined on  $\mathcal{A}$ , such that  $(X_\theta, \sigma)$  is infinite. Then*

1.  $h < |\mathcal{A}|$ ,
2. *there is a primitive length  $r$  substitution shift  $(X_{\theta'}, \sigma')$  of height one such that  $(X_\theta, \sigma) \cong (X_{\theta'} \times \{0, \dots, h-1\}, T)$  where*

$$T(x, i) := \begin{cases} (x, i+1) & \text{if } 0 \leq i < h-1 \\ (\sigma'(x), 0) & \text{if } i = h-1 \end{cases}$$

Furthermore  $h$ ,  $\theta'$ , and the conjugacy between  $(X_\theta, \sigma)$  and  $(X_{\theta'} \times \{0, \dots, h-1\}, T)$  can be determined algorithmically.

The substitution  $\theta'$  described in Proposition 3.4 is called a *pure base* of  $\theta$ .

For every  $\Psi \in \text{Aut}(X_{\theta'}, \sigma')$  and  $0 \leq i \leq h-1$  we can define  $\Psi_i \in \text{Aut}(X_{\theta'} \times \{0, \dots, h-1\}, T)$  as

$$\Psi_i(x, j) := \begin{cases} (\Psi(x), j+i) & \text{if } j+i < h \\ (\Psi(\sigma'(x)), j+i \bmod h) & \text{if } j+i \geq h. \end{cases}$$

**Proposition 3.5.** *Let  $\theta$  be a primitive, length  $r$  substitution, of height  $h$ , and such that  $(X_\theta, \sigma)$  is infinite. Let  $\theta'$  be a pure base of  $\theta$ . Then  $\text{Aut}(X_\theta, \sigma) = \{\Psi_j : \Psi \in \text{Aut}(X_{\theta'}, \sigma') \text{ and } 0 \leq j < h\}$ .*

*Proof.* By Proposition 3.4,  $(X_\theta, \sigma)$  is conjugate to  $(X_{\theta'} \times \{0, \dots, h-1\}, T)$ , and so  $\text{Aut}(X_\theta, \sigma)$  is group isomorphic to  $\text{Aut}(X_{\theta'} \times \{0, \dots, h-1\}, T)$ . By the definition of  $\Psi_i$ , we see that  $\{\Psi_j : \Psi \in \text{Aut}(X_{\theta'}, \sigma') \text{ and } 0 \leq j < h\} \subset \text{Aut}(X_{\theta'} \times \{0, \dots, h-1\}, T)$ .

Conversely, given  $\Psi \in \text{Aut}(X_{\theta'} \times \{0, \dots, h-1\}, T)$ , suppose that  $\Psi(x, 0) \in X_{\theta'} \times \{i\}$  for some  $x \in X_{\theta'}$ . Define  $f(x) = \pi_2(\Psi(x, 0))$ , where  $\pi_2$  is the projection onto the second coordinate and notice that  $f$  is a continuous  $\sigma'$ -invariant function on  $X_{\theta'}$  and hence is constant. This, and the fact  $\Psi$  commutes with  $T$  implies that  $\Psi = \Psi'_i$  for some  $\Psi' \in \text{Aut}(X_{\theta'}, \sigma')$ .  $\square$

Proposition 3.5 tells us that in order to compute the automorphism group of a primitive constant-length substitution, or, as we shall see later, the set of conjugacies between two such substitutions, it is sufficient to work with with a pure base  $\theta'$  of  $\theta$ . Thus, apart from the statements of our main results Theorems 3.22 and 3.27, and Corollary 3.7, we henceforth assume that our substitutions are of height one.

Let  $\theta$  be a length  $r$  substitution. We write  $\theta(a) = \theta_0(a) \dots \theta_{r-1}(a)$ ; with this notation we see that for each  $0 \leq i \leq r-1$ , we have a map  $\theta_i : \mathcal{A} \rightarrow \mathcal{A}$  where  $\theta_i(a)$  is the  $(i+1)$ -st letter of  $\theta(a)$ .

Let  $\theta$  have pure base  $\theta'$ . We say that  $\theta$  has *column number*  $c$  if for some  $k \in \mathbb{N}$ , and some  $(i_1, \dots, i_k)$ ,  $|\theta'_{i_1} \circ \dots \circ \theta'_{i_k}(\mathcal{A})| = c$  and  $c$  is the least such number. In particular, if  $\theta$  has column number one, then we will say that  $\theta$  has a *coincidence*. If for each  $0 \leq i \leq r-1$  and  $a \neq b$ ,  $\theta_i(a) \neq \theta_i(b)$ , we say  $\theta$  is *bijective*. Henceforth we shall be working with primitive substitutions  $\theta$  which are length  $r$  and such that  $(X_\theta, \sigma)$  is infinite.

The following is shown by Dekking [9], with partial results by Kamae [18] and Martin [20].

**Theorem 3.6.** *Let  $\theta$  be a primitive, length  $r$  substitution, of height one, and such that  $(X_\theta, \sigma)$  is infinite. Then the maximal equicontinuous factor of  $(X_\theta, \sigma)$  is  $(\mathbb{Z}_r, +1)$ .*

Recall the map  $\kappa$  as defined in Theorem 3.3.

**Corollary 3.7.** *Let  $\theta$  be a primitive, length  $r$  substitution, such that  $(X_\theta, \sigma)$  is infinite.*

1. *Let  $\theta$  have height one. If  $\kappa$  is injective (in particular if  $\theta$  has a coincidence), then  $\text{Aut}(X_\theta, \sigma)$  is cyclic.*
2. *Let  $\theta$  have height  $h > 1$  and  $\theta'$  be the pure base of  $\theta$ . Suppose that  $\kappa : X_{\theta'} \rightarrow \mathbb{Z}_r$  is injective, so that  $\text{Aut}(X_{\theta'}, \sigma') = \langle \Phi \rangle$  with  $\kappa(\Phi) = 1/k$ . Then  $\text{Aut}(X_\theta, \sigma)$  is abelian and generated by  $T$  and  $\Phi_0$  (in the above notation). Further,  $\text{Aut}(X_\theta, \sigma)$  is cyclic if and only if  $\gcd(k, h) = 1$ .*

*Proof.* Let  $\theta$  be as in the statement, and first suppose that  $\theta$  has height one, so that its maximal equicontinuous factor is  $(\mathbb{Z}_r, +1)$ . By the discussion in Section 2.3.1 and Theorem 2.3,  $\text{Aut}(X_\theta, \sigma)$  is finitely generated. Let a set of generators be  $\Phi_0 = \sigma, \Phi_1, \dots, \Phi_n$ , say. Then Theorem 2.3 implies  $\kappa(\Phi_0) = 1$  and  $\kappa(\Phi_i) = p_i/q_i$  for each  $i$  for some  $p_i \in \mathbb{Z}$  and  $q_i \geq 1$ . By Lemma 3.2,  $\kappa(\text{Aut}(X_\theta, \sigma))$  is cyclic. The result follows from the injectivity of  $\kappa$ . In particular, since substitutions with coincidences are precisely those whose shifts are a somewhere one-to-one extension of their maximal equicontinuous factor [9], we deduce by Theorem 2.3 that if  $\theta$  has a coincidence, then  $\text{Aut}(X_\theta, \sigma)$  is cyclic.

If  $\theta$  has height  $h > 1$ , let  $\theta'$  be the pure base of  $\theta$ . We assume that  $\theta'$  has a coincidence. By the above,  $\text{Aut}(X_{\theta'}, \sigma')$  is cyclic, and generated by some  $\Phi$  with  $\kappa(\Phi) = 1/k$  for some  $k \in \mathbb{N}$ . Suppose that  $\gcd(h, k) = 1$  and let  $ah + bk = 1$  with  $0 \leq b < h$ . Then consider the automorphism  $\Psi$  of  $X_{\theta'} \times \{0, 1, \dots, h-1\}$  given by  $\Psi(x, i) = (\Phi^a(x), i+b)$  if  $i+b < h$  or  $(\Phi^a(\sigma'(x)), i+b-h)$  otherwise (that is  $\Psi = (\Phi^a)_b$  in the previous notation). Define an equivalence relation on  $X_{\theta'} \times \mathbb{Z}$  to be the transitive closure of  $(\sigma'(x), i) \sim (x, i+h)$  so that  $X_{\theta'} \times \{0, \dots, h-1\}$  is a system of representatives of the equivalence classes. In this notation,  $\Psi(x, i) = (\Phi^a(x), i+b)$ . Notice that  $\Psi^k(x, i) = (\Phi^{ak}(x), i+kb) = (\sigma'^{a_k}x, i+kb) \sim (x, i+kb+ah) = (x, i+1) = T(x, i)$ , so that  $T \in \langle \Psi \rangle$ . Since  $\gcd(a, k) = 1$ , some power of  $\Phi^a$  is of the form  $\Phi \circ \sigma'^m$ . Hence  $\Phi_0 \in \langle \Psi \rangle$  also, so that  $\text{Aut}(X_\theta, \sigma)$  is cyclic as required.

Conversely if  $\gcd(h, k) = d > 1$ , then let  $\Psi = \Phi_0^{k/d} \circ T^{-h/d}$ . We have  $\Psi \neq \text{Id}$ , but  $\Psi^d = \text{Id}$ . Since infinite cyclic groups are torsion-free, we see that  $\text{Aut}(X_\theta, \sigma)$  is not cyclic.  $\square$

Substitutions for which  $\gcd(h, k) > 1$  can be constructed. For, starting with a (primitive, nonperiodic) substitution with a coincidence on two letters, its automorphism group equals  $\mathbb{Z}$  [3]. We can use the techniques in Example 3.23 to obtain a substitution shift  $(X_\theta, \sigma)$ , the  $\kappa$  values of whose automorphism group is  $\langle \frac{1}{k} \rangle$ . Now building a tower of height  $h$  over  $(X_\theta, \sigma)$  gives us the desired substitution shift. We remark also that Part (1) of Corollary 3.7 holds, with essentially the same proof, for linearly recurrent Toeplitz systems [11] whose maximal equicontinuous factor is  $\mathbb{Z}_r$ .

A constant-length substitution is called *injective* if  $\theta(i) \neq \theta(j)$  for any distinct  $i$  and  $j$  in  $\mathcal{A}$ . It will be convenient to deal with injective substitutions in what follows. The following theorem of Blanchard, Durand and Maass allows us to restrict our attention to that situation.

**Theorem 3.8** ([15]). *Let  $\theta$  be a constant-length substitution such that  $X_\theta$  is infinite. Then there exists an injective substitution  $\theta'$  such that  $(X_\theta, \sigma)$  is topologically conjugate to  $(X_{\theta'}, \sigma)$ . Further, the conjugacy and its inverse are explicitly computable, so that  $\text{Aut}(X_\theta, \sigma)$  can be determined algorithmically from  $\text{Aut}(X_{\theta'}, \sigma)$ .*

*Proof.* We only sketch the proof that the inverse of the conjugacy is computable, as this is not described in [15]. There, we see that  $(X_\theta, \sigma)$  is conjugate to  $(X_{\theta'}, \sigma)$  via  $\tau$ , which is a composition of less than  $|\mathcal{A}|$  algorithmically-determined letter-to-letter maps,  $\tau = \tau_1 \circ \dots \circ \tau_N$ . By iteration, it suffices to show that a single one of these maps has a computable inverse, so we assume that  $N = 1$  and  $\tau_1 = \tau$ .

By definition,  $\theta \circ \tau = \theta$ , so that to explicitly describe  $\tau^{-1}$ , we only need explicitly describe  $\theta^{-1}$ . If  $\theta$  is a constant-length substitution on  $\mathcal{A}$ , then there exists  $k \leq |\mathcal{A}|^2$  such that if  $\theta^k(a) = \theta^k(b)$ , then  $\theta^{k-1}(a) = \theta^{k-1}(b)$ . Fix any letter  $a$ . The bilateral *recognizability* of  $\theta^k$  [22, Definition 1.1 and Theorem 3.1] implies that there exists an  $\ell \geq k$  such that  $\theta^\ell(a)$  does not appear starting at the interior of any  $\theta^k$

word. Such an  $\ell$  can be obtained by a simple algorithm. Uniform recurrence of  $(X_\theta, \sigma)$  implies that if  $L$  is large enough, then any word of length  $L$  in  $\mathcal{L}_{X_\theta}$  contains some  $\theta^\ell(a)$  as a subword. Also, this  $L$  is computable [13, Proposition 25]. Thus  $\theta^{-1}$  can be explicitly described and the proof of our claim is complete.  $\square$

### 3.4 The description of $\{z \in \mathbb{Z}_r : |\pi^{-1}(z)| > c\}$

We continue to assume that  $\theta$  is a length  $r$  substitution of height one, so that  $\mathbb{Z}_r$  is the maximal equicontinuous factor of  $(X_\theta, \sigma)$ . We let  $c$  be the column number of  $\theta$ . We show that all points of  $\mathbb{Z}_r$  have at least  $c$  preimages and most points of  $\mathbb{Z}_r$  have exactly  $c$  preimages. We study the structure of the subset of  $\mathbb{Z}_r$  having excess preimages.

If  $P_0 := \theta^n(X_\theta)$ , then  $P_0$  generates a  $\sigma^{r^n}$ -cyclic partition of size  $r^n$  [9, Lemma II.7]. We use the notation of [18], [9], using  $\Lambda_{r^n}$  to denote the equivalence relation whose classes are the members of this cyclic  $\sigma^{r^n}$ -partition, i.e. we write  $\Lambda_{r^n}(x) = i$  if  $x \in \sigma^i(P_0)$ .

Using the maps  $\Lambda_n$ , we can define a maximal equicontinuous factor map  $\pi : X_\theta \rightarrow \mathbb{Z}_r$ . Note that if  $\Lambda_n(x) = i$ , then  $\Lambda_{n+1}(x) \equiv i \pmod{r^n}$ . Using this we can define  $\pi(x) := \dots x_2 x_1 x_0$  where for each  $n \in \mathbb{N}$ ,  $\Lambda_n(x) = \sum_{i=0}^{n-1} r^i x_i$ .

**Definition 1.** Let  $\theta$  be a length  $r$  substitution with column number  $c$  and of height one. Let  $\Sigma_\theta$  be the one-sided shift on  $\{0, 1, \dots, r-1\}$ , whose points are left-infinite, and whose set of forbidden words is

$$\mathcal{F}_\theta := \{w = w_k \dots w_1 \in (\mathbb{Z}/r\mathbb{Z})^+ : |\theta_{w_1} \circ \dots \circ \theta_{w_k}(\mathcal{A})| = c\}.$$

Note that  $\mathcal{F}_\theta \neq \emptyset$ . It can happen that  $\Sigma_\theta = \emptyset$ : this is the case for the Thue-Morse substitution:  $\theta(0) = 01, \theta(1) = 10$ .

**Lemma 3.9.** Let  $\theta$  be a primitive, length  $r$  substitution on  $\mathcal{A}$  with column number  $c$ , of height one, and such that  $(X_\theta, \sigma)$  is infinite. Then  $\Sigma_\theta$  is either empty or a sofic shift.

*Proof.* Let  $S = \{\mathcal{A}\} \cup \{\theta_{w_1} \circ \dots \circ \theta_{w_k}(\mathcal{A}) : k \geq 1; w_1, \dots, w_k \in \{0, 1, \dots, r-1\}\}$  and  $V = \{A \in S : |A| > c\}$ . This is the vertex set of a directed labelled graph. If  $A \in V$  and  $\theta_i(A) \in V$ , then we add an edge from  $\theta_i(A)$  to  $A$  labelled with the symbol  $i$ . We claim that the set  $\Sigma_\theta$  is the set of left-infinite one-sided sequences in  $(\mathbb{Z}/r\mathbb{Z})^\mathbb{N}$  such that for every  $m < n$  and finite segment,  $w_{n-1}w_{n-2} \dots w_m$ , the sequence  $(w_m, w_{m+1}, \dots, w_{n-1})$  is the sequence of labels of a path in the graph. To see this, notice that  $(w_m, w_{m+1}, \dots, w_{n-1})$  is the sequence of labels of a path in the graph if and only if there are elements  $A_m, A_{m+1}, \dots, A_n$  of  $V$  such that  $\theta_{w_k}(A_{k+1}) = A_k$  for each  $m \leq k < n$ , so that  $|\theta_{w_m} \circ \theta_{w_{m+1}} \circ \dots \circ \theta_{w_{n-1}}(A_n)| = |A_m| > c$  and hence  $|\theta_{w_m} \circ \theta_{w_{m+1}} \circ \dots \circ \theta_{w_{n-1}}(\mathcal{A})| > c$ .

The reason for the reversal of the order of the indices is that we want to maintain compatibility with standard definitions of sofic:  $\Sigma_\theta$  is a collection of left-infinite sequences, which is precisely the collection of edge-labellings of infinite paths in the graph listed in reverse order.  $\square$

**Example 3.10.** [3]. If  $|\mathcal{A}| = 2$ , then any primitive length  $r$  substitution generating an infinite shift has height one, by Proposition 3.4, and for each  $i$ ,  $\theta_i(\mathcal{A}) = \mathcal{A}$  or  $|\theta_i(\mathcal{A})| = 1$ . If  $\mathcal{C}$  is the set of indices  $i$  such that  $|\theta_i(\mathcal{A})| = 1$ , we see that  $\mathcal{F}_\theta = \{w = w_1 \dots w_k \in (\mathbb{Z}/r\mathbb{Z})^+ : w_i \in \mathcal{C} \text{ for some } i \in \{1, \dots, k\}\}$ . In this case  $\Sigma_\theta$  is the full shift on the alphabet  $(\mathbb{Z}/r\mathbb{Z}) \setminus \mathcal{C}$ .

**Example 3.11.** Let  $\mathcal{A} = \{a, b, c\}$ , and let  $\theta(a) = abbc$ ,  $\theta(b) = cbab$  and  $\theta(c) = cbba$ ;  $\theta$  has a coincidence and is of height one. The labelled graph that generates  $\Sigma_\theta$  is shown in Figure 1.

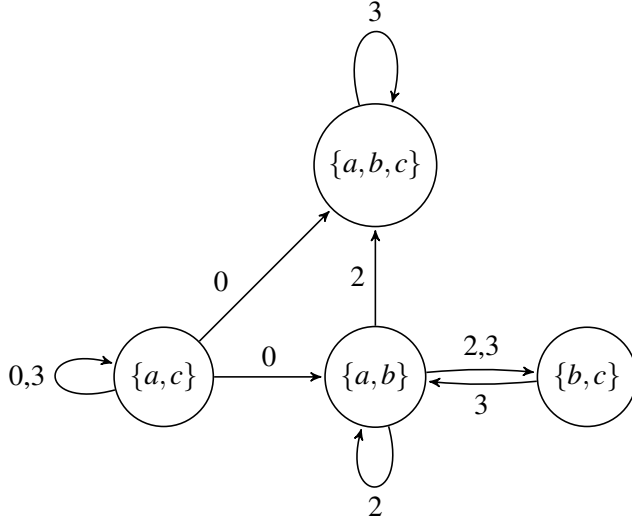


Figure 1: The graph for Example 3.11.

Depending on  $\theta$ , we may have to slightly alter the sofic shift that we work with. We write  $\bar{a}$  to denote  $\dots aaa$ .

**Definition 2.** Let  $\theta$  be a length  $r$  substitution with column number  $c$ . Let  $\Sigma_\theta$  be as above. Let  $P = \{x \in X_\theta : \theta^n(x) = x \text{ for some } n\}$ , that is the set of (two-sided) periodic points under the substitution.

$$\hat{\Sigma}_\theta := \begin{cases} \Sigma_\theta \cup \{\bar{0}, \overline{r-1}\} & \text{if } |P| > c \\ \Sigma_\theta & \text{otherwise;} \end{cases}$$

if  $\Sigma_\theta$  is sofic, then  $\hat{\Sigma}_\theta$  is sofic. We shall show below (in Corollary 3.13) that  $\hat{\Sigma}_\theta$  is not empty and hence by the definition above,  $\hat{\Sigma}_\theta$  is always sofic. Define

$$\tilde{\Sigma}_\theta = \{x : \sigma^n(x) \in \hat{\Sigma}_\theta \text{ for some } n \geq 0\}.$$

**Lemma 3.12.** Let  $\theta$  be a primitive, length  $r$  substitution, with column number  $c$ , of height one and such that  $(X_\theta, \sigma)$  is infinite. Then  $\tilde{\Sigma}_\theta = \{z \in \mathbb{Z}_r : |\pi^{-1}(z)| > c\}$ .

*Proof.* We separate the proof into two parts:  $z \in \mathbb{Z}$  and  $z \notin \mathbb{Z}$ .

First suppose  $z \in \mathbb{Z}$  and  $|\pi^{-1}(z)| > c$ . Then there are more than  $c$   $\theta$ -periodic points, so that  $z \in \mathbb{Z} \subset \tilde{\Sigma}_\theta$ .

For the converse, notice that if either  $0^j \notin \mathcal{F}_\theta$  for all  $j$  or  $(r-1)^j \notin \mathcal{F}_\theta$  for all  $j$ , then there are more than  $c$  one-sided right- or left-  $\theta$ -periodic points respectively. By primitivity, each one-sided  $\theta$ -periodic point extends to at least one two-sided  $\theta$ -periodic point, so that we deduce that if  $\mathbb{Z} \cap \tilde{\Sigma}_\theta$  is non-empty then there are more than  $c$   $\theta$ -periodic points. Hence if  $z \in \mathbb{Z} \cap \tilde{\Sigma}_\theta$ , we see  $|\pi^{-1}(z)| > c$ .

If  $z \in \tilde{\Sigma}_\theta \setminus \mathbb{Z}$ , then there exists  $n_0$  such that for all  $n \geq n_0$ ,  $z_n \dots z_{n_0} \notin \mathcal{F}_\theta$ . We may assume without loss of generality (since  $|\pi^{-1}(z)| = |\pi^{-1}(z+k)|$  for any  $k \in \mathbb{Z}$ ) that  $z_{n_0-1}, \dots, z_0$  are all 0. Let  $z' = \sigma^{n_0}z$ , so that for all  $n \geq 0$ ,  $z'_{n-1} \dots z'_0 \notin \mathcal{F}_\theta$ . This means that for each  $n$  there are distinct letters  $\bar{a}_n^{(1)}, \dots, \bar{a}_n^{(c+1)}$ , and letters  $a_n^{(1)}, \dots, a_n^{(c+1)}$  such that  $\theta_{z'_0} \circ \dots \circ \theta_{z'_{n-1}}(a_n^{(i)}) = \bar{a}_n^{(i)}$  for  $1 \leq i \leq c+1$ . By passing to a subsequence we can assume that  $(a_n^{(1)}, \dots, a_n^{(c+1)}) = (a_1, \dots, a_{c+1})$  and  $(\bar{a}_n^{(1)}, \dots, \bar{a}_n^{(c+1)}) = (\bar{a}_1, \dots, \bar{a}_{c+1})$  for all  $n$ .

For  $i = 1, \dots, c+1$ , take  $x^{(n),i} \in X$  such that for each  $i$ ,  $x_0^{(n),i} = \bar{a}_i$ ,  $\pi(x^{(n),i})$  ends with  $z'_{n-1} \dots z'_0$  and the segment of  $x^{(n),i}$  from coordinate index  $-\sum_{i=0}^{n-1} z'_i r^i$  to  $r^n - 1 - \sum_{i=0}^{n-1} z'_i r^i$  agrees with  $\theta^n(a_i)$ . Taking limits gives at least  $c+1$  distinct points of  $\pi^{-1}(z')$ . Since  $\theta$  is injective and  $\theta^{n_0}x \in \pi^{-1}(z)$  for any  $x \in \pi^{-1}(z')$ , applying  $\theta^{n_0}$  to these points shows  $|\pi^{-1}(z)| \geq c+1$ .

If  $z \notin \tilde{\Sigma}_\theta \cup \mathbb{Z}$ , then for infinitely many  $n$ , there exist  $k_n$  such that  $z_{n+k_n} \dots z_n \in \mathcal{F}_\theta$ . Let  $x = (x_j)$  satisfy  $\pi(x) = z$  and fix  $n$ . Since  $|\theta_{z_n} \circ \dots \circ \theta_{z_{n+k_n}}(\mathcal{A})| = c$ , there are  $c$  possible choices for the block of  $x_j$ 's for  $j \in [0, r^n - 1] - \sum_{i=0}^{n-1} r^i z_i$ . Since  $z \notin \mathbb{Z}$ , the union of these intervals exhausts all of  $\mathbb{Z}$ .  $\square$

**Corollary 3.13.** *Let  $\theta$  be a primitive, length  $r$  substitution defined on  $\mathcal{A}$  with column number  $c$ , of height one, and such that  $(X_\theta, \sigma)$  is infinite. Then the function  $|\pi^{-1}(z)| : \mathbb{Z}_r \rightarrow \mathbb{N}$  has minimum value  $c$ , is non-constant, and  $\hat{\Sigma}_\theta$  is a nonempty and proper subset of  $\mathbb{Z}_r$ .*

*Proof.* Let  $z \in \mathbb{Z}_r$  and for each  $a \in \mathcal{A}$ , let  $x^{(a)} \in X_\theta$  satisfy  $x_0^{(a)} = a$ . Let  $y^{(n,a)} = \sigma^{m_n} \theta^n(x^{(a)})$ , where  $m_n = z_{n-1}r^{n-1} + \dots + z_1r + z_0$ . Let  $Y^{(n)} = \{y^{(n,a)} : a \in \mathcal{A}\}$  and let  $Y^{(\infty)}$  be the set of accumulation points of  $Y^{(n)}$  as  $n \rightarrow \infty$ . The 0th coordinates of the  $y^{(n,a)}$  exhaust  $\theta_{z_0} \circ \theta_{z_1} \circ \dots \circ \theta_{z_{n-1}}(\mathcal{A})$ , so that  $|Y^{(n)}| \geq c$ , and also  $|Y^{(\infty)}| \geq c$ . Since  $Y^{(\infty)}$  is a subset of  $\pi^{-1}(z)$ , we conclude  $|\pi^{-1}(z)| \geq c$  for each  $z \in \mathbb{Z}_r$ .

Since  $\mathcal{F}_\theta \neq \emptyset$ , we take a word  $w \in \mathcal{F}_\theta$ , and then a point  $z \in \mathbb{Z}_r \setminus \mathbb{Z}$  which contains  $w$  infinitely often. Then  $z \notin \tilde{\Sigma}_\theta$ , and by Lemma 3.12,  $|\pi^{-1}(z)| = c$ .

To see that there exist points  $z$  with  $|\pi^{-1}(z)| > c$ , we note that our shifts will always have at least one non-trivial right asymptotic orbit equivalence class. This follows from the fact that any continuous, positively expansive map on an infinite compact metric space cannot be invertible [7]. We pick  $x$  and  $x'$  that are right asymptotic, and we suppose that  $x_n = x'_n$  for  $n \geq 0$ . Let  $z = \pi(x) = \pi(x')$ . The sets  $\theta_{z_0} \circ \theta_{z_1} \circ \dots \circ \theta_{z_n}(\mathcal{A})$  are decreasing in  $n$  and so equal  $\mathcal{A}_0$ , a set of cardinality at least  $c$ , for all large  $n$ . For each  $a \in \mathcal{A}_0$ , there is a  $y$  in  $\pi^{-1}z$  with  $y_0 = a$ . As  $x$  and  $x'$  are two elements in  $\pi^{-1}z$  with the same 0 coordinate, we conclude that  $|\pi^{-1}z| \geq c+1$ .  $\square$

### 3.5 Bounding denominators of $\kappa(\Phi)$

We have established that if  $\theta$  has column number  $c$ , then the set  $\{z : |\pi^{-1}(z)| > c\}$  is the set of points whose tail lies in a sofic shift that is a proper subshift of the full one-sided shift on  $r$  letters, that is  $\tilde{\Sigma}_\theta$ . Although we already know that  $\kappa(\Phi)$  is rational for  $\Phi \in \text{Aut}(X_\theta, \sigma)$ , Part (2) of Theorem 3.3 tells us that  $\tilde{\Sigma}_\theta \subset \tilde{\Sigma}_\theta + \kappa(\Phi)$ , which will allow us to give bounds on the denominator of  $\kappa(\Phi)$ . We also recover a special case of a result in [13], namely that for our shifts, all endomorphisms are automorphisms.

We write  $d$  for the  $r$ -adic metric:  $d(x, y) = r^{-k}$ , where  $k = \min\{j : x_j \neq y_j\}$  (or  $d(x, y) = 0$  if  $x = y$ ). We also need a metric on the circle. We identify the circle with  $[0, 1)$  and define  $d_\circ(x, y) = \min_{n \in \mathbb{Z}} |n + (x - y)|$ .

If  $X$  is a subshift of  $\mathbb{Z}_r$ , we say that  $x$  has a *tail* belonging to  $X$  if there exists  $k \in \mathbb{N}$  such that  $\sigma^k(x) \in X$ .



**Lemma 3.14.** *Let  $X$  be a proper sofic shift of  $\mathbb{Z}_r$ , so that there is a word of length  $j$  that does not occur in the language of  $X$ . Suppose that there exists  $t \in \mathbb{Z}_r$  such that  $x+t$  has a tail belonging to  $X$  for each  $x \in X$ . Then  $nt \in \mathbb{Z}$  for some  $n$  satisfying  $\gcd(n, r) = 1$  and  $n \leq r^j - 1$ .*

**Lemma 3.15.** *Let  $\Omega$  be an infinite subset of  $S^1$ . Then for each  $\varepsilon > 0$ , there exists an  $N$  such that  $N \star \Omega := \{n\omega : 1 \leq n \leq N; \omega \in \Omega\}$  is  $\varepsilon$ -dense.*

*Proof.* If  $\Omega$  contains an irrational point, then its multiples are dense and we're done. Otherwise if  $\Omega \subset \mathbb{Q}$ , given  $\varepsilon$ , since  $\Omega$  is infinite, it must contain a point  $\frac{p}{q}$  (in lowest terms) with  $q > 1/\varepsilon$ . Now  $\{j \cdot \frac{p}{q} \bmod 1 : j \in \mathbb{N}_0\}$  exhausts  $\{\frac{i}{q} : 0 \leq i < q\}$ , so that  $q \star \Omega$  is  $\varepsilon$ -dense.  $\square$

*Proof of Lemma 3.14.* Define  $\psi_n : \mathbb{Z}_r \rightarrow S^1$  by  $\psi_n(x) = \sum_{k=0}^{n-1} x_k r^{k-n}$ . For  $\omega \in S^1$ , we denote  $r\omega \bmod 1$  by  $rs$ . Let  $\Omega(t) \subset S^1$  be the set of limit points of  $\{\psi_n(t) : n \in \mathbb{N}_0\}$ . Notice that if  $\psi_{n_i}(t) \rightarrow \omega$ , then  $\psi_{n_i-1}(t) \rightarrow r\omega$ , so that  $\Omega(t)$  is closed under multiplication by  $r$ . Also  $\Omega(mt) = m\Omega(t)$  and  $\Omega(m+t) = \Omega(t)$  for  $m \in \mathbb{Z}$ .

We first claim that if  $t$  is not rational, or if  $t$  is rational with a denominator exceeding  $r^j$ , then there exists an  $m$  such that  $\Omega(mt)$  contains an element of  $S^1$  in  $(r^{-j}/2, r^{-j})$ .

Suppose first that  $\Omega(t)$  is a finite set. Since it is closed under multiplication by  $r$ , it must consist of rationals (given  $\omega \in \Omega(t)$ , there exist  $0 \leq m < n \leq |\Omega(t)|$  such that  $r^m \omega$  and  $r^n \omega$  agree modulo 1, so that  $(r^n - r^m)\omega \in \mathbb{Z}$ ). In particular, by taking a least common multiple, there exists a  $Q \in \mathbb{N}_0$  such that  $Q\omega = 0 \bmod 1$  for all  $\omega \in \Omega(t)$ . Now as  $d_\circ(\psi_n(t), \Omega(t)) \rightarrow 0$ , we have  $d_\circ(\psi_n(Qt), 0) = d_\circ(Q\psi_n(t), Q\Omega(t)) \rightarrow 0$ . This implies  $Qt \in \mathbb{Z}$ , so that this is the rational case. By assumption, the denominator of  $t$  exceeds  $r^j$ . By Lemma 3.1, we can write  $t$  as  $p/q$  with  $\gcd(p, q) = \gcd(q, r) = 1$ . Then there exist  $a, m \in \mathbb{Z}$  such that  $mt = a - 1/q$  in  $\mathbb{Z}_r$ . Now  $\Omega(mt) = \Omega(-1/q)$ . Since  $q$  is coprime to  $r$ , there exists a  $b$  such that  $r^b \equiv 1 \pmod{q}$  (or  $q|r^b - 1$ ). Write  $r^b - 1 = sq$  and notice that the expansion of  $-1/q = s/(1 - r^b)$  in  $\mathbb{Z}_r$  equals  $\bar{s} = \dots sss$  where  $\bar{s}$  has period  $b$ . Now we see  $\psi_{bn}(-1/q) \rightarrow s/(r^b - 1) = 1/q$ , so that  $1/q \in \Omega(mt)$ . There exists an  $l \in \mathbb{Z}$  such that  $l/q \in (r^{-j}/2, r^{-j})$ . In particular, we have shown the claim above (for  $\Omega(lmt)$ ) in the case that  $t$  is rational.

If  $\Omega(t)$  is infinite, then by Lemma 3.15, there is an  $M$  such that  $\bigcup_{m=1}^M \Omega(mt)$  is  $1/(2r^j + 1)$ -dense. Hence there is an  $m \leq M$  such that  $\Omega(mt) \cap (r^{-j}/2, r^{-j}) \neq \emptyset$  completing the proof of the claim.

There is a natural bijection  $\Phi$  between  $j$ -blocks of  $X$  and  $\mathbb{Z}/r^j\mathbb{Z}$ : given a  $j$ -block  $B = [x_{j-1} \dots x_0]$ ,  $\Phi(B) = x_{j-1}r^{j-1} + \dots + x_0$ . (Recall that we are writing elements of  $\mathbb{Z}_r$  from right to left so that  $x_{j-1}$  is the “most significant digit” of  $B$ ).

Hence for a  $j$ -block  $B$ , there is a natural notion of its successor  $B + 1$ . Let  $B$  be chosen so that  $B$  is a valid  $j$ -block, but  $B + 1$  is not. To show that such a  $B$  exists, list the  $j$ -blocks sequentially, each one the successor of the previous one. At some point in the sequence there is a valid  $j$ -block followed by an invalid one (otherwise all would be valid or all would be invalid). Since  $X$  is sofic, it has dense periodic points. Let  $z$  be a periodic point in  $X$  containing  $B$ 's. Let  $p$  be the period of  $z$ .

By the above, there exist  $\omega \in \Omega$  and  $m$  such that  $\frac{1}{2}r^{-j} < m\omega \bmod 1 < r^{-j}$ . Since  $\omega \in \Omega$ , there exists an increasing sequence  $(n_i)$  such that  $\psi_{n_i}(t) \rightarrow \omega$ . For all sufficiently large  $i$ ,  $\frac{1}{2}r^{-j} < \psi_{n_i}(mt) < r^{-j}$ . Refine the sequence  $(n_i)$  such that  $\frac{1}{2}r^{-j} < \psi_{n_i}(mt) < r^{-j}$  for each  $i$  and all of the  $n_i$  are congruent modulo  $p$  (such a subsequence exists by the pigeonhole principle: at least one congruence class must contain infinitely many terms of the original sequence).

Now let  $k$  be chosen so that the block  $B$  appears in the coordinate range  $n_i - j$  to  $n_i - 1$  for each  $i$  in  $w := \sigma^k(z)$ . Let  $\alpha = \Phi(B)/r^j$ . The fact that  $B$  appears in these blocks is equivalent to the assertion that  $\psi_{n_i}(w) \in [\alpha, \alpha + r^{-j}]$ , while the appearance of  $B + 1$  in locations  $n - j$  to  $n - 1$  of  $x \in X$  is equivalent to the assertion that  $\psi_n(x) \in [\alpha + r^{-j}, \alpha + 2r^{-j}]$ .

Now notice that for each  $i$ , either  $\psi_{n_i}(w + mt)$  or  $\psi_{n_i}(w + 2mt)$  lies in  $[\alpha + r^{-j}, \alpha + 2r^{-j}]$ . In particular, one of  $w + mt$  and  $w + 2mt$  contains infinitely many  $B + 1$  blocks, and hence does not have a tail lying in  $X$ . Hence we have shown that if  $t$  is irrational, or  $t$  is rational with denominator exceeding  $r^j$ , then the hypotheses of the theorem cannot be satisfied.  $\square$

**Theorem 3.16.** *Let  $\theta$  be a primitive, length  $r$  substitution on  $A$ , with column number  $c$ , of height one, and such that  $(X_\theta, \sigma)$  is infinite. Then  $\text{End}(X_\theta, \sigma) = \text{Aut}(X_\theta, \sigma)$ .*

*If  $\mathcal{F}_\theta$  contains a word of length  $j$ , then any  $\Phi \in \text{Aut}(X_\theta, \sigma)$  satisfies  $\kappa(\Phi) = \ell/n$  for some  $n \leq r^j - 1$  with  $\gcd(n, r) = 1$ , and some  $\ell \in \mathbb{Z}$ , and  $\Phi^{nk} = \sigma^{\ell k}$  for some  $1 \leq k \leq c$ .*

*Proof.* Let  $\theta$  be a substitution as in the statement and let  $\Sigma_\theta$ ,  $\hat{\Sigma}_\theta$  and  $\tilde{\Sigma}_\theta$  be as constructed above. Let  $\Phi$  be an endomorphism of  $X_\theta$  and let  $t = \kappa(\Phi)$ . By Theorem 3.3, Lemma 3.12 and Corollary 3.13, we have  $t + \tilde{\Sigma}_\theta \subset \tilde{\Sigma}_\theta$ . By Lemma 3.14, we deduce  $t$  is rational,  $\ell/n$  say, with denominator coprime to  $r$  and at most  $r^j - 1$ .

Now  $\kappa(\Phi^n \sigma^{-\ell}) = 0$ , so that  $\Phi^n \sigma^{-\ell}$  is a self-map of  $\pi^{-1}(z)$  for any  $z$  in  $\mathbb{Z}_r$ . Choosing  $z$  so that  $|\pi^{-1}(z)| = c$ , we see that there is a  $1 \leq k \leq c$  such that  $(\Phi^n \sigma^{-\ell})^k$  has a fixed point. By minimality of  $X_\theta$ ,  $\Phi^{nk} = \sigma^{\ell k}$  and we see that  $\Phi$  is bijective, so  $\Phi \in \text{Aut}(X_\theta, \sigma)$ .  $\square$

**Example 3.17.** *We continue with Example 3.11, which we already noted has a coincidence and is of height one. Since the word 1 belongs to  $\mathcal{F}_\theta$ , Theorem 3.16 implies for any  $\Phi \in \text{Aut}(X_\theta)$ ,  $\kappa(\Phi)$  has denominator 1 or 3. If there were an automorphism with denominator 3, then by taking powers and composing with a power of the shift, we could find an automorphism  $\Phi$  with  $\kappa(\Phi) = -1/3 = \bar{1}$ . However, from the proof of Theorem 3.16, this would imply  $\bar{1} + \tilde{\Sigma}_\theta \subset \tilde{\Sigma}_\theta$ , which is false as  $\bar{0} \in \tilde{\Sigma}_\theta$ , but  $\bar{1} \notin \tilde{\Sigma}_\theta$ . Hence  $\kappa(\Phi) \in \mathbb{Z}$  for all  $\Phi \in \text{Aut}(X_\theta, \sigma)$  and such  $\Phi$  are powers of the shift by the theorem.*

Recall that if  $(X_\theta, \sigma)$  is a two-sided shift, then  $(\bar{X}_\theta, \bar{\sigma})$  is the corresponding one-sided shift.

**Theorem 3.18.** *Let  $\theta$  be a primitive, length  $r$  substitution, of height one, and such that  $(X_\theta, \sigma)$  is infinite. If  $\kappa$  is injective (in particular if  $\theta$  has a coincidence), then  $\text{Aut}(\bar{X}_\theta, \bar{\sigma}) = \{\text{Id}\}$ .*

*Proof.* Any automorphism  $\bar{\Phi}$  of  $(\bar{X}_\theta, \bar{\sigma})$  gives rise to an endomorphism  $\Phi$  of  $(X_\theta, \sigma)$ , which satisfies  $\Phi^n = \sigma^k$  for some  $k, n \in \mathbb{N} \cup \{0\}$ . As  $\bar{\Phi}$  is an automorphism, we have  $k = 0$  and so  $\kappa(\Phi) = 0$ . Now the injectivity of  $\kappa$  implies that  $\Phi = \text{Id}$ , so  $\bar{\Phi} = \text{Id}$  also.  $\square$

### 3.6 Computing $\text{Aut}(X_\theta, \sigma)$

In this section, we use the bounds on the denominator appearing in  $\kappa$  together with bounds on the radius of the block code to obtain an algorithm to compute the automorphism group of a constant-length substitution.

Let  $\Phi \in \text{Aut}(X_\theta, \sigma)$ . By the Curtis–Hedlund–Lyndon theorem, there exist  $l, r$ , and a map  $f: \mathcal{A}^{l+r+1} \rightarrow \mathcal{A}$  with the property that  $(\Phi(x))_n = f(x_{n-l}, \dots, x_n, \dots, x_{n+r})$  for all  $x$  and  $n$ . Let  $l, r$  be the smallest possible integers so that such an  $f$  exists. We call  $l, r$  the left and right radius of  $\Phi$  respectively. We say  $\Phi$  has radius  $R$  if its left radius and right radius are both at most  $R$ .

The recognizability of  $\theta$  implies that any  $x \in X_\theta$  can be written in a unique way as  $x = \sigma^k(\theta(y))$  where  $y \in X_\theta$  and, if  $\theta$  is length  $r$ ,  $0 \leq k < r$  (see [16] for a proof in the injective case, which is all we need). The following proposition tells us that up to a shift, every automorphism has a small radius, parallel to the case of *reduced* constant-length substitutions with partial continuous spectrum in the work of Host and Parreau [17, Theorem 1.3]. We remark also that Coven, Dykstra, Keane and LeMasurier have a similar result in [5, Theorem 1].

**Proposition 3.19.** *Let  $\theta$  be an injective primitive, length  $r$  substitution of height one, and such that  $(X_\theta, \sigma)$  is infinite. If  $\Phi \in \text{Aut}(X_\theta, \sigma)$  has the property that  $\kappa(\Phi) \in \mathbb{Z}_r \setminus \mathbb{Z}$  is periodic, then  $\Phi$  has right radius zero and left radius at most 1.*

*Further, if  $\kappa(\Phi) = k/(1 - r^p)$  for  $0 < k < r^p - 1$ , one has*

$$\theta^{-c!p} \circ \sigma^{-k(1+r^p+r^{2p}+\dots+r^{(c!-1)p})} \circ \Phi \circ \theta^{c!p} = \Phi,$$

where  $c$  is the column number.

*If  $\kappa(\Phi) = 0$ , then  $\Phi$  has left and right radius at most 1 and  $\Phi$  satisfies  $\theta^{-c!} \circ \Phi \circ \theta^{c!} = \Phi$ .*

*Proof.* We first assume that  $\theta$  has a coincidence, so that  $\kappa$  is an injection. Let  $R$  be the radius of  $\Phi$  and let  $t = \kappa(\Phi)$ . By assumption,  $t$  is periodic as a point of  $\mathbb{Z}_r$ , with period  $p$ , say. Further, the repeating block is not all 0's or all  $r-1$ 's. Hence there exists  $n$ , a multiple of  $p$ , such that  $t = r^n t + w$  and  $R < w < r^n - R$ . Now  $(\Phi(x))_i$  is determined by  $x_{[i-R, i+R]}$ , so that  $(\sigma^{-w} \circ \Phi(x))_i$  is determined by  $x_{[i-w-R, i-w+R]}$  and in particular,  $(\sigma^{-w} \circ \Phi(x))_{[0, r^n]}$  is determined by  $x_{[-r^n, r^n]}$ .

Since  $\kappa(\sigma^{-w} \circ \Phi) \in r^n \mathbb{Z}_r$ ,  $\sigma^{-w} \circ \Phi$  maps  $\theta^n(X_\theta)$  to itself, and since recognizability tells us that  $\theta^n: X_\theta \rightarrow \theta^n(X_\theta)$  is invertible, let  $\Psi(y) := \theta^{-n} \circ \sigma^{-w} \circ \Phi \circ \theta^n(y)$ . The above ensures that  $\Psi$  is an automorphism of  $(X, \sigma)$ , and the injectivity of  $\theta$  implies that  $(\Psi(y))_0$  depends on  $y_{[-1, 0]}$ . Notice that  $\pi(\theta(x)) = r\pi(x)$ , so that  $\pi(\Phi \circ \theta^n x) = r^n \pi(x) + \kappa(\Phi)$  and  $\pi(\Psi(x)) = \pi(\theta^{-n} \circ \sigma^{-w} \circ \Phi \circ \theta^n x) = r^{-n}(-w + r^n \pi(x) + \kappa(\Phi)) = \pi(x) + t$ . Hence  $\kappa(\Psi) = t = \kappa(\Phi)$ . By injectivity of  $\kappa$  we see that  $\Phi = \Psi$ , so that  $\Phi$  has left radius 1 and right radius 0 as required.

Suppose now that  $\theta$  has column number  $c$  and  $t \in \kappa(\text{Aut}(X_\theta, T)) \setminus \mathbb{Z}$ . Theorem 3.3 tells us that the group homomorphism  $\kappa$  has kernel of size  $K \leq c$ . Let  $\{\Phi_1, \dots, \Phi_K\} = \kappa^{-1}(t)$ . As above, we find  $n$  and  $w$  that work for all of the  $\Phi_i$ 's, and consider the map which sends  $\Phi \in \{\Phi_1, \dots, \Phi_K\}$  to  $\theta^{-n} \circ \sigma^{-w} \circ \Phi \circ \theta^n$ . This map is a bijection of  $\{\Phi_1, \dots, \Phi_K\}$ , and, since  $\theta^{-n} \circ \sigma^{-w} \circ \Phi \circ \theta^n$  has left radius one and right radius 0, we see that all the maps  $\Phi_i$  satisfy this property.

If  $t = k/(1 - r^p)$ , the calculation above shows that  $\Phi \mapsto \theta^{-p} \circ \sigma^{-k} \circ \Phi \circ \theta^p$  is a permutation of  $\kappa^{-1}(t)$ . Since  $|\kappa^{-1}(t)| \leq c$ , one sees by iterating

$$\theta^{-c!p} \circ \sigma^{-k(1+r^p+r^{2p}+\dots+r^{(c!-1)p})} \circ \Phi \circ \theta^{c!p} = \Phi.$$

The properties of  $\Phi$  such that  $\kappa(\Phi) = 0$  are established similarly. □

If  $t \in \mathbb{Z}_r$  is such that  $nt =: k \in \mathbb{Z}$  for some  $n$ , then we define  $\lfloor t \rfloor = \lfloor k/n \rfloor \in \mathbb{Z}$ . We define  $\lceil t \rceil$  similarly as  $\lceil k/n \rceil$ . We can now rephrase the radius of  $\Phi$  in terms of  $\kappa(\Phi)$ .

**Corollary 3.20.** *Let  $\theta$  be an injective, primitive, length  $r$  substitution of height one, such that  $(X_\theta, \sigma)$  is infinite. Then for each  $\Phi \in \text{Aut}(X_\theta, \sigma)$  such that  $\kappa(\Phi) \notin \mathbb{Z}$ , one has that  $\Phi(x)_0$  depends only on  $x_{\lfloor \kappa(\Phi) \rfloor, \lceil \kappa(\Phi) \rceil}$ .*

*Proof.* Let  $n$  be chosen so that  $\kappa(\Phi) - n$  is periodic in  $\mathbb{Z}_r$  with some period  $p$ , and let  $\Psi = \Phi \circ \sigma^{-n}$ . If the periodic block is the base  $r$  representation of a number  $a$ , we have  $\kappa(\Psi) = a/(1 - r^p)$  with  $0 < a < r^p - 1$ , so that  $(r^p - 1)\kappa(\Psi) = -a$ . Hence  $\kappa(\Phi) = n - a/(r^p - 1)$ . In particular,  $\lfloor \kappa(\Phi) \rfloor = n - 1$  and  $\lceil \kappa(\Phi) \rceil = n$ . By Proposition 3.19,  $\Psi(x)_0$  depends on  $x_{[-1, 0]}$ , and the result follows.  $\square$

We remark that a version of Corollary 3.20 exists for integral values of  $\kappa(\Phi)$  also. If  $\theta$  has a coincidence, then integral values of  $\kappa(\Phi)$  only arise from powers of the shift. Otherwise, if  $\kappa(\Phi) = n$ , then  $\Phi(x)_0$  is determined by  $x_{n-1}$ ,  $x_n$  and  $x_{n+1}$ . If  $\theta$  has the property that for each pair  $a$  and  $a'$  of distinct elements of  $\mathcal{A}$ , there is a  $k$  such that  $\theta^k(a)$  and  $\theta^k(a')$  differ in some coordinate that is neither the first nor the last of the block, then one can prove that  $\Phi(x)_0$  is determined by  $x_n$ . A natural question is whether this is the case for all primitive constant-length substitutions.

Given left and right radii  $m$  and  $n$  and a map  $f: \mathcal{A}^{m+n+1} \rightarrow \mathcal{A}$ , we write  $\Phi_{m,n}^f$  for the map  $\mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$  given by  $\Phi_{m,n}^f(x)_i = f(x_{i-m}, \dots, x_{i+n})$ .

**Proposition 3.21.** *Let  $\theta$  be an injective primitive, length  $r$  substitution on  $\mathcal{A}$ , with column number  $c$ , of height one, and such that  $(X_\theta, \sigma)$  is infinite. Let  $p \geq 1$  and  $0 < k < r^p - 1$ .*

*Let  $f: \mathcal{A}^2 \rightarrow \mathcal{A}$ . Then  $\Phi := \Phi_{1,0}^f$  satisfies  $\Phi \in \text{Aut}(X_\theta, \sigma)$  and  $\kappa(\Phi) = k/(1 - r^p)$  if and only if:*

1. *if  $x_0x_1x_2 \in \mathcal{L}_{X_\theta}$ , then  $f(x_0, x_1)f(x_1, x_2) \in \mathcal{L}_{X_\theta}$ ; and*
2. *For each  $x_{-1}x_0x_1 \in \mathcal{L}(X_\theta)$  and  $0 \leq i < r^{c!p}$*

$$f(\theta^{c!p}(x_{-1} \cdot x_0)_{i-1}, \theta^{c!p}(x_{-1} \cdot x_0)_i) = \theta^{c!p}(f(x_{-1}, x_0)f(x_0, x_1))_{N+i},$$

*where  $N = k(1 + r^p + \dots + r^{(c!-1)p})$  and  $x_{-1} \cdot x_0$  denotes a finite segment of a bi-infinite sequence with  $x_{-1}$  in the  $-1$ st coordinate and  $x_0$  in the  $0$ th coordinate.*

*Similarly given  $g: \mathcal{A}^3 \rightarrow \mathcal{A}$ ,  $\Phi_{1,1}^g$  satisfies  $\Phi_{1,1}^g \in \text{Aut}(X_\theta, \sigma)$  and  $\kappa(\Phi_{1,1}^g) = 0$  if and only if*

3.  *$g(x_0, x_1, x_2)g(x_1, x_2, x_3) \in \mathcal{L}_{X_\theta}$  whenever  $x_0x_1x_2x_3 \in \mathcal{L}_{X_\theta}$ ; and*
4. *For any  $x_{-1}x_0x_1 \in \mathcal{L}_{X_\theta}$  and  $0 \leq i < r^{c!}$ ,*

$$g(u_{i-1}, u_i, u_{i+1}) = (\theta^{c!}g(x_{-1}, x_0, x_1))_i,$$

*where  $u = u_{-r^{c!}}u_{-r^{c!}+1} \dots u_{-1} \cdot u_0 \dots u_{2r^{c!}-1}$  is the word obtained by applying  $\theta^{c!}$  to the word  $x_{-1} \cdot x_0x_1$ .*

*Proof.* First suppose that there exists  $\Phi \in \text{Aut}(X_\theta, \sigma)$  such that  $\kappa(\Phi) = t := k/(1 - r^p)$ . By Proposition 3.19,  $\Phi = \Phi_{1,0}^f$  for a map  $f: \mathcal{A}^2 \rightarrow \mathcal{A}$ . Condition (1) is obviously satisfied and condition (2) is the commutation relation of Proposition 3.19 expressed in terms of  $f$ .

Conversely, suppose  $f$  satisfies Conditions (1) and (2). Let  $\Phi = \Phi_{1,0}^f$ . Condition (2) implies  $\Phi(\theta^{c!p}x)_i = (\sigma^N \circ \theta^{c!p} \circ \Phi(x))_i$  for  $0 \leq i < r^{c!p}$ . The same equation applied to iterates of  $x$  under  $\sigma$  yields  $\Phi \circ \theta^{c!p} = \sigma^N \circ \theta^{c!p} \circ \Phi$ . Iterating this equality and using the relation  $\sigma^r \circ \theta = \theta \circ \sigma$  gives

$$\Phi \circ \theta^{nc!p} = (\sigma^N \circ \theta^{c!p})^n \circ \Phi = \sigma^{N(1+r^{c!p}+\dots+r^{(n-1)c!p})} \circ \theta^{nc!p} \circ \Phi.$$

We let  $x \in X_\theta$  and show that  $\Phi(x) \in X_\theta$ . For any  $n$ , we can write  $x = \sigma^{-j} \circ \theta^{npc!}(y)$  for some  $y \in X_\theta$  and  $0 \leq j < r^{npc!}$ . Now applying the above, we have

$$\Phi(x) = \sigma^{-j} \circ \Phi(\theta^{npc!}(y)) = \sigma^l(\theta^{npc!}(\Phi(y))),$$

where

$$l = -j + N(1 + r^{c!p} + \dots + r^{(n-1)c!p}).$$

Since by (1) all 2-words of  $\Phi(y)$  belong to  $\mathcal{L}_{X_\theta}$ , we deduce all subwords of  $\theta^{npc!}(\Phi(y))$  of size  $1 + r^{npc!}$  belong to  $\mathcal{L}_{X_\theta}$ . Since  $n$  is arbitrary, we deduce  $\Phi(x) \in X_\theta$ .

Since  $\Phi$  is shift-commuting by construction, we have  $\Phi \in \text{End}(X_\theta, \sigma)$  and hence  $\Phi \in \text{Aut}(X_\theta, \sigma)$  by Theorem 3.16. In particular,  $t := \kappa(\Phi)$  is now defined. Now  $\pi(\Phi(\theta^{pc!}(x))) = t + r^{pc!}\pi(x)$  and  $\pi(\sigma^N(\theta^{pc!}(\Phi(x)))) = k(1 + r^p + \dots + r^{(c!-1)p}) + r^{pc!}(\pi(x) + t)$ . Since these agree, we have  $t(1 - r^p) = k$  as required.

The argument that Conditions (3) and (4) characterize elements of  $\ker \kappa$  is similar.  $\square$

We can now deduce

**Theorem 3.22.** *Let  $\theta$  be a primitive, length  $r$  substitution on  $\mathcal{A}$  such that  $(X_\theta, \sigma)$  is infinite. Then there is an algorithm to compute  $\text{Aut}(X_\theta, \sigma)$ .*

*Proof.* If  $\theta$  does not have height one, we use Proposition 3.4 to compute a pure base of  $\theta$ . Proposition 3.5 tells us how to retrieve  $\text{Aut}(X_\theta, \sigma)$  from the automorphism group of its pure base. Similarly, if the pure base is not injective, we use Theorem 3.8 to compute its automorphism group from the automorphism group of a conjugate injective substitution shift. Hence it suffices to give an algorithm for injective substitutions of height one.

We can algorithmically compute the column number,  $c$  and also  $j$ , the length of the shortest word in  $\mathcal{F}_\theta$  from the transition graphs described in Section 3.4. Recall from Lemma 3.2 that  $\kappa(\text{Aut}(X_\theta, T))$  is a cyclic subgroup of  $\mathbb{Z}_r$  generated by  $1/d$  for some integer  $d$ . By Theorem 3.16,  $d$  satisfies  $\gcd(d, r) = 1$  and  $d \leq r^j - 1$ . Now  $\text{Aut}(X_\theta, T)$  is the semi-direct product of  $\ker \kappa$  with the cyclic subgroup generated by any element of  $\kappa^{-1}(-1/d)$ .

To find  $\ker \kappa$ , one applies Proposition 3.21 to test all of the finitely many 3-block maps. Note that one can algorithmically list the words of length two and three that belong to  $\mathcal{L}_{X_\theta}$ . One then needs to find an element of  $\kappa^{-1}(-1/d)$  for the largest  $d$  so that this set is non-empty. By the above, there are finitely many values of  $d$  to check. Since  $\gcd(d, r) = 1$ , we see  $d|r^p - 1$  for some  $p$  and then  $-1/d$  can be expressed as  $k/(1 - r^p)$  for some  $0 < k < r^p - 1$ . By Proposition 3.21, there are finitely many 2-block maps to check for each potential  $d$ .  $\square$

We do not address the question of efficiency.

From Corollary 3.7, we know that for substitutions with a coincidence,  $\text{Aut}(X_\theta, \sigma)/\langle \sigma \rangle$  is a finite cyclic group. In Example 3.17, this group was trivial, and it is natural to wonder if this is always the case. The following example shows that the quotient can indeed be non-trivial.

**Example 3.23.** Let  $\mathcal{A} = \{a, b, c\}$ ,  $\theta(a) = aba$ ,  $\theta(b) = cba$  and  $\theta(c) = ccb$ . This example appears in Dekking's article [9] just after Theorem 14. He describes how  $(X_\theta, \sigma)$  is a relabelling of the second power shift of the substitution  $\theta'$  where  $\theta'(0) = 011$  and  $\theta'(1) = 101$ . That is, the shift with symbols  $\boxed{01}$ ,  $\boxed{11}$  and  $\boxed{10}$  with  $\theta(\boxed{01}) = \boxed{01}\boxed{11}\boxed{01}$ ,  $\theta(\boxed{11}) = \boxed{10}\boxed{11}\boxed{01}$  and  $\theta(\boxed{10}) = \boxed{10}\boxed{10}\boxed{11}$ . Then  $(X_\theta, \sigma)$  is conjugate to  $(X_{\theta'}, \sigma'^2)$  (where we use  $\sigma'$  to denote the shift on  $X_{\theta'}$ ). As a consequence, the map  $\sigma'$  is conjugate to an automorphism  $\Phi$  of  $(X_\theta, \theta)$  satisfying  $\Phi^2 = \sigma$ . Using the techniques we have developed, it can be verified that  $t = \bar{1}$  is the only non-trivial periodic 3-adic integer such that  $\tilde{\Sigma}_\theta + t = \tilde{\Sigma}_\theta$ . Since  $\Phi^2 = \sigma$ , we have  $\kappa(\Phi) = \bar{1}2$  and we have shown that  $\text{Aut}(X_\theta, \sigma) = \langle \Phi \rangle$ .

### 3.7 Computing $\text{Conj}(X_\theta, X_{\theta'})$ .

The techniques of Section 3.6 can be generalized to compute the set of conjugacies between two substitution shifts generated by primitive constant-length substitutions, one of which generates an infinite shift. We use  $\kappa$  to also denote the restriction to  $\text{Conj}(X_\theta, X_{\theta'})$  of the map in Theorem 3.3. We continue to assume that our maximal equicontinuous factor mappings are those that were defined in Section 3.4, so that  $r^n | \pi(x)$  if and only if  $x \in \theta^n(X_\theta)$ .

**Proposition 3.24.** Let  $\theta$  and  $\theta'$  be primitive, length  $r$  substitutions, both with column number  $c$  and of height one, and such that  $(X_\theta, \sigma)$  and  $(X_{\theta'}, \sigma)$  are infinite. Suppose that  $\Phi \in \text{Conj}(X_\theta, X_{\theta'})$ . Then  $\kappa(\Phi)$  is rational, and if  $\mathcal{F}_\theta$  contains a word of length  $j$ , then  $n\kappa(\Phi) \in \mathbb{Z}$  for some  $0 \leq n \leq (r-1)(r^j - 1)$ .

*Proof.* Let  $\pi : X_\theta \rightarrow \mathbb{Z}_r$  and  $\pi' : X_{\theta'} \rightarrow \mathbb{Z}_r$  denote the relevant maximal equicontinuous factor mappings, and suppose that  $\Phi \in \text{Conj}(X_\theta, X_{\theta'})$ . Since  $\kappa(\Phi \circ \sigma^n) = \kappa(\Phi) + n$ , we can assume, by composing  $\Phi$  with a power of the shift if necessary, that  $\kappa(\Phi) \equiv 0 \pmod{r}$ , i.e. that  $\kappa(\Phi) \in r\mathbb{Z}_r$ . In this case, noting that  $\pi(\theta(x)) = r\pi(x)$ , we have  $\pi'(\Phi(\theta(x))) = \kappa(\Phi) + \pi(\theta(x)) = \kappa(\Phi) + r\pi(x) \equiv 0 \pmod{r}$ . Since recognizability tells us that  $\theta : X_\theta \rightarrow \theta(X_\theta)$  and  $\theta' : X_{\theta'} \rightarrow \theta'(X_{\theta'})$  are invertible, then  $\Psi := \theta'^{-1} \circ \Phi \circ \theta$  is a well defined conjugacy. Also,

$$\pi(x) + \kappa(\Psi) = \pi'(\theta'^{-1} \circ \Phi \circ \theta(x)) = \frac{1}{r}(\pi(\theta(x)) + \kappa(\Phi)) = \pi(x) + \frac{\kappa(\Phi)}{r},$$

so that  $\kappa(\Psi) = \kappa(\Phi)/r$ . Finally, (1) of Theorem 3.3 tells us that

$$\kappa(\Psi \circ \Phi^{-1}) = \kappa(\Psi) - \kappa(\Phi) = \kappa(\Phi)\left(\frac{1}{r} - 1\right),$$

and since  $\Psi \circ \Phi^{-1} \in \text{Aut}(X_{\theta'}, \sigma)$ , so that  $\kappa(\Psi \circ \Phi^{-1})$  is rational, we deduce our first claim. By Theorem 3.16,  $n\kappa(\Psi \circ \Phi^{-1}) \in \mathbb{Z}$  for some  $0 \leq n \leq r^j - 1$ , and our second claim follows.  $\square$

The proofs of the following two propositions are a straightforward generalization of those of Propositions 3.19 and 3.21.



**Proposition 3.25.** *Let  $\theta$  and  $\theta'$  be injective primitive, length  $r$  substitutions, both with column number  $c$  and of height one, and such that  $(X_\theta, \sigma)$  and  $(X_{\theta'}, \sigma)$  are infinite. If  $\Phi \in \text{Conj}(X_\theta, X_{\theta'})$  has the property that  $\kappa(\Phi) \in \mathbb{Z}_r \setminus \mathbb{Z}$  is periodic, then  $\Phi$  has right radius zero and left radius at most 1. Further, one has*

$$\theta'^{-c!p} \circ \sigma^{-k(1+r^p+r^{2p}+\dots+r^{(c!-1)p})} \circ \Phi \circ \theta^{c!p} = \Phi,$$

where  $c$  is the column number and  $\kappa(\Phi) = k/(1-r^p)$  with  $0 < k < r^p - 1$ .

If  $\kappa(\Phi) = 0$ , then  $\Phi$  has left and right radius at most 1, and

$$\theta'^{-c!} \circ \Phi \circ \theta^{c!} = \Phi.$$

**Proposition 3.26.** *Let  $\theta$  and  $\theta'$  be injective primitive, length  $r$  substitutions on  $\mathcal{A}$  and  $\mathcal{A}'$  respectively, both with column number  $c$  and of height one, and such that  $(X_\theta, \sigma)$  and  $(X_{\theta'}, \sigma)$  are infinite. Let  $p \geq 1$  and suppose that  $0 < k < r^p - 1$ .*

*Let  $f: \mathcal{A}^2 \rightarrow \mathcal{A}'$ . Then  $\Phi := \Phi_{1,0}^f$  satisfies  $\Phi \in \text{Conj}(X_\theta, X_{\theta'})$  and  $\kappa(\Phi) = k/(1-r^p)$  if and only if:*

1. *if  $x_0x_1x_2 \in \mathcal{L}_{X_\theta}$ , then  $f(x_0, x_1)f(x_1, x_2) \in \mathcal{L}_{X_{\theta'}}$ ; and*
2. *For each  $x_{-1}x_0x_1 \in \mathcal{L}(X_\theta)$  and  $0 \leq i < r^{c!p}$*

$$f(\theta^{c!p}(x_{-1} \cdot x_0)_{i-1}, \theta^{c!p}(x_{-1} \cdot x_0)_i) = \theta'^{c!p}(f(x_{-1}, x_0)f(x_0, x_1))_{N+i},$$

where  $N = k(1+r^p+\dots+r^{(c!-1)p})$  and  $x_{-1} \cdot x_0$  denotes a finite segment of a bi-infinite sequence with  $x_{-1}$  in the  $-1$ st coordinate and  $x_0$  in the  $0$ th coordinate.

Similarly given  $g: \mathcal{A}^3 \rightarrow \mathcal{A}'$ ,  $\Phi_{1,1}^g$  satisfies  $\Phi_{1,1}^g \in \text{Conj}(X_\theta, X_{\theta'})$  and  $\kappa(\Phi_{1,1}^g) = 0$  if and only if

3.  *$g(x_0, x_1, x_2)g(x_1, x_2, x_3) \in \mathcal{L}_{X_{\theta'}}$  whenever  $x_0x_1x_2x_3 \in \mathcal{L}_{X_\theta}$ ; and*
4. *For any  $x_{-1}x_0x_1 \in \mathcal{L}_{X_\theta}$  and  $0 \leq i < r^{c!}$ ,*

$$g(u_{i-1}, u_i, u_{i+1}) = (\theta'^{c!}g(x_{-1}, x_0, x_1))_i,$$

where  $u$  is the block of length  $3r^{c!}$  given by  $u_{-r^{c!}}u_{-r^{c!}+1}\dots u_{-1} \cdot u_0 \dots u_{2r^{c!}-1} = \theta^{c!}(x_{-1} \cdot x_0x_1)$ .

Let  $\theta'$  be a primitive length  $r$  substitution with column number  $c$ , such that  $(X_{\theta'}, \sigma)$  is infinite. Let  $\theta$  be another constant-length substitution. If  $(X_\theta, \sigma)$  and  $(X_{\theta'}, \sigma)$  were topologically conjugate, then both systems would be infinite, and both would have the same maximal equicontinuous factor. Thus, conjugacy of the two systems implies that  $\theta$  and  $\theta'$  have the same height, and the appropriate dynamical formulation of Cobham's theorem [6] tells us that the lengths of  $\theta$  and  $\theta'$  are powers of the same integer. Corollary 3.13 implies that both have column number  $c$ . By taking a power of  $\theta'$  or  $\theta$  if necessary, and by considering the pure bases of our two substitutions, we are in a position to apply Propositions 3.24, 3.25 and 3.26, to conclude:

**Theorem 3.27.** *Let  $\theta$  and  $\theta'$  be primitive, constant-length substitutions. Suppose further  $(X_\theta, \sigma)$  is infinite. Then there is an algorithm to decide whether  $(X_\theta, \sigma)$  and  $(X_{\theta'}, \sigma)$  are topologically conjugate. In the case that they are topologically conjugate, the algorithm yields a topological conjugacy and hence one can algorithmically determine  $\text{Conj}(X_\theta, X_{\theta'})$ .*



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